Lecture 3: Finite $(\infty, 1)$-limits; fibration categories

Homotopical semantics of type theory

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1 Limits in quasicategories

Last lecture, we defined what a quasicategory is and what it means for an object of a quasicategory to be terminal. Now we'll use the definition of terminal object to define what it means for an object, or rather a cone, to be a limit of a diagram in a quasicategory. You can get more details about the material in this section from Joyal [Joy02, §4].

Recall that a terminal object in X is a limit of the unique diagram in X indexed by the empty category. In 1-category theory, we can use the concept of terminal object to define limits of other shapes. Namely, for a diagram $d\colon \mathcal{K} \to \mathcal{C}$ in some category $\mathcal{C}$, we can define a *limit cone for* d to be a terminal object of the *category* $\mathcal{C}/d$ *of cones over* d, whose

- objects are pairs of $c \in \mathcal{C}$ with natural transformations $\alpha\colon \operatorname{const} c \to d$ from the constant diagram on c;
- morphisms $f\colon (c, \alpha) \to (c', \alpha')$ are morphisms $f\colon c \to c'$ such that $\alpha' \circ \operatorname{const} f = \alpha$.

In the case $\mathcal{K} = 1$, which is not a very interesting diagram to take a limit over, the category of cones over $d\colon 1 \to \mathcal{C}$ is the slice category over d.

To define limits in a quasicategory X, then, we will first define the *quasicategory of cones in* X over a given diagram. As a useful side effect, we'll get a definition of *slice quasicategory*.

Definition 1.1. The *augmented simplex category* Δ_{a} is the category of finite linear orders and monotone functions between them.

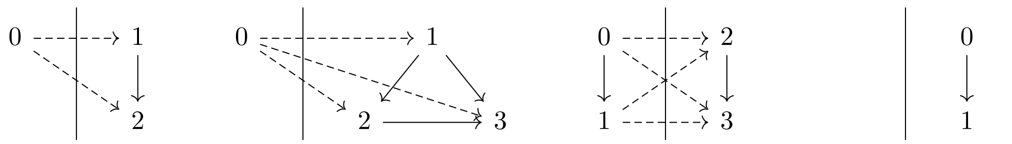
Recall that the simplex category Δ is the category of *inhabited* finite linear orders, so there is a fully faithful inclusion $i\colon \Delta \to \Delta_{\mathrm{a}}$. For $[n] \in \Delta$, we also write $[n]$ for its image in Δ_{a}. We write $[-1] \coloneqq \emptyset \in \Delta_{\mathrm{a}}$ for the empty linear order, the one element of Δ_{a} that does not belong to Δ.

We write $\mathbf{sSet}_{\mathrm{a}} \coloneqq \mathrm{PSh}(\Delta_{\mathrm{a}})$ for the category of presheaves on Δ_{a}, and $\Delta^n_{\mathrm{a}} \coloneqq \text{よ}[n]$ for $n \geq -1$. As always, the functor $i\colon \Delta \to \Delta_{\mathrm{a}}$ induces a triple of adjoint functors

$$\mathbf{sSet} \; \underset{i_*}{\overset{i_!}{\rightleftarrows}} \; \mathbf{sSet}_{\mathrm{a}}, \qquad i_! \dashv i^* \dashv i_*, \quad i^*\colon \mathbf{sSet}_{\mathrm{a}} \to \mathbf{sSet}$$

The restriction functor $i^*\colon \mathbf{sSet}_{\mathrm{a}} \to \mathbf{sSet}$ sends $X \in \mathbf{sSet}_{\mathrm{a}}$ to the simplicial set $(i^*X)_n \coloneqq X_n$, simply forgetting the set X_{-1}. Its right adjoint, the *terminal augmentation*, sends $X \in \mathbf{sSet}$ to $i_*X \in \mathbf{sSet}$ with $(i_*X)_{-1} \coloneqq 1$ and $(i_*X)_n \coloneqq X_n$ for $n \geq 1$. In particular, one can check that it preserves simplices: we have $i_*\Delta^n \cong \Delta^n$ for $n \geq 0$.

Definition 1.2. Define the *join* of $[m], [n] \in \Delta_{\mathrm{a}}$ to be $[m] \star [n] \coloneqq [m + 1 + n] \in \Delta_{\mathrm{a}}$. This extends to a functor $(- \star -)\colon \Delta_{\mathrm{a}} \times \Delta_{\mathrm{a}} \to \Delta_{\mathrm{a}}$.



(a) $[0] \star [1] \cong [2]$ (b) $[0] \star [2] \cong [3]$ (c) $[1] \star [1] \cong [3]$ (d) $[-1] \star [1] \cong [1]$

Figure 1: Examples of joins of simplics

The join of two augmented simplices $[m]$ and $[n]$ can be drawn in a suggestive way as the complete bipartite graph on m and n, as shown in Figure 1. Note how the join $[0] \star [n]$ forms a cone over $[n]$, while $[1] \star [n]$ can be read as a morphism of cones over the same copy of $[n]$.

We extend the join to simplicial sets, using augmented simplicial sets as an intermediary.

Definition 1.3. We define the *join of augmented simplicial sets* $(- \star -)\colon \mathbf{sSet}_{\mathrm{a}} \times \mathbf{sSet}_{\mathrm{a}} \to \mathbf{sSet}_{\mathrm{a}}$ to be the unique colimit-preserving extension of the join of augmented simplices along the Yoneda embedding in both variables:

$$\begin{array}{ccc} \Delta_{\mathrm{a}} \times \Delta_{\mathrm{a}} & \xrightarrow{\;- \star -\;} & \Delta_{\mathrm{a}} \\ {\scriptstyle \text{よ} \times \text{よ}}\downarrow & & \downarrow{\scriptstyle \text{よ}} \\ \mathbf{sSet}_{\mathrm{a}} \times \mathbf{sSet}_{\mathrm{a}} & \xrightarrow[- \star -]{} & \mathbf{sSet}_{\mathrm{a}}. \end{array}$$

In other words, we define $X \star Y$ to be the colimit of the joins of each augmented simplex in X with each augmented simplex in Y:

$$X \star Y \cong \operatorname*{colim}_{\substack{[m] \in \Delta_{\mathrm{a}}, f\colon \Delta^m_{\mathrm{a}} \to X \\ [n] \in \Delta_{\mathrm{a}}, g\colon \Delta^n_{\mathrm{a}} \to Y}} \Delta^m_{\mathrm{a}} \star \Delta^n_{\mathrm{a}}$$

Then we define the *join of simplicial sets* $(- \star -)\colon \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$ by composing with the terminal augmentation and restriction:

$$\mathbf{sSet} \times \mathbf{sSet} \xrightarrow{\;i_* \times i_*\;} \mathbf{sSet}_{\mathrm{a}} \times \mathbf{sSet}_{\mathrm{a}} \xrightarrow{\;- \star -\;} \mathbf{sSet}_{\mathrm{a}} \xrightarrow{\;i^*\;} \mathbf{sSet}$$

Because $i_* \Delta^n \cong \Delta^n_{\mathrm{a}}$ and $i^* \Delta^n_{\mathrm{a}} \cong \Delta^n$, we also have $\Delta^m \star \Delta^n \cong \Delta^{m+1+n}$ in $\mathbf{sSet}$. We can also calculate:

Proposition 1.4. For $X \in \mathbf{sSet}$, $\emptyset \star X \cong X \star \emptyset \cong X$.

As a consequence of this and the functoriality of the join, we have maps $\upsilon_0\colon X \cong X \star \emptyset \to X \star Y$ and $\upsilon_1\colon Y \cong \emptyset \star Y \to X \star Y$ from the two components into the join.

When $X \in \mathbf{sSet}$ is a quasicategory, we can think of $\Delta^0 \star X$ as adjoining a new initial object (as Figure 1 suggests), and respectively $X \star \Delta^0$ as adjoining a new terminal object. (We see here that, despite the notation, the join is not symmetric.)

Now let us fix some $K \in \mathbf{sSet}$ that we will think of as a diagram shape, so that functors $d\colon K \to X$ are "diagrams of shape K in X". A *cone over d* will be an extension fitting in the diagram

$$\begin{array}{ccc} & K & \\ {\scriptstyle \upsilon_1} \swarrow & & \searrow {\scriptstyle d} \\ \Delta^0 \star K & \xrightarrow[c]{} & X \end{array}$$

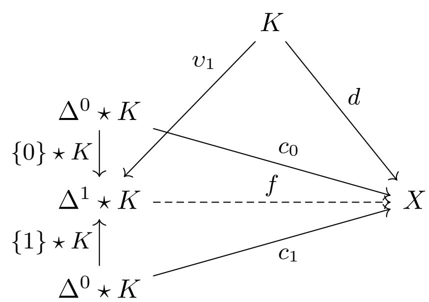
with the point $cv_0 \colon \Delta^0 \to X$ acting as its apex.

To define what a terminal cone is, we need the collection of cones on d to assemble into a quasicategory—that is, we need to define what morphisms, 2-cells, etc. of cones should be.

Definition 1.5. For $d \colon K \to X$, define $X/d \in \mathbf{sSet}$ by

$$(X/d)([n]) \coloneqq (K/\mathbf{sSet})((\Delta^n \star K, v_1), (X, d)) = \left\{ \begin{array}{ccc} & K & \\ {}^{v_1}\swarrow & & \searrow^{d} \\ \Delta^n \star K & \overset{}{\underset{c}{\dashrightarrow}} & X \end{array} \right\}.$$

For example, a morphism of cones will be a functor $f \colon \Delta^1 \star K \to X$ under K, and its domain and codomain are given by the simplicial maps: given two cones $c_0, c_1 \colon \Delta^0 \star K \to X$, we have $f \colon c_0 \to c_1$ when f fits into the following diagram.



Consider again how the picture in Figure 1c can be read as a morphism between two cones over the diagram $2 \to 3$.

One can check that the functor $\mathbf{sSet} \to K/\mathbf{sSet}$ sending $A \in \mathbf{sSet}$ to $v_1 \colon K \to A \star K$ preserves all colimits. It then follows essentially by definition that the functor $K/\mathbf{sSet} \to \mathbf{sSet}$ that sends $d \colon K \to X$ to X/d is its right adjoint. With this characterization, we often reduce theorems about cones to properties of the join.

Proposition 1.6 ([Joy02, Corollary 3.9])**.** If X is a quasicategory, then X/d is a quasicategory for every simplicial set K and functor $d \colon K \to X$.

Finally we can define:

Definition 1.7. For $K \in \mathbf{sSet}$, X a quasicategory, and $d \colon K \to X$, a cone $c \colon \Delta^0 \star K \to X$ under K is *limiting* if it is a terminal object of X/d.

Definition 1.8. A quasicategory X *has finite limits* if every diagram $d \colon K \to X$ where K has finitely many non-degenerate cells admits a limiting cone. A functor $F \colon X \to Y$ between quasicategories *preserves finite limits* if it sends finite limiting cones to finite limiting cones.

For example, a *pullback* can defined as the limit of a diagram $d \colon \Lambda^2_2 \to X$, recalling that the (2,2)-horn has the shape of a pullback cospan:

$$\begin{array}{ccc} & & 1 \\ & & \downarrow \\ 0 & \longrightarrow & 2. \end{array}$$

It is a non-obvious but true fact that, as in 1-category theory, a quasicategory has finite limits if and only if it has a terminal object and pullbacks [Cis19, Theorem 7.3.27].

2 Fibration categories

So far, we have been working with the collection of quasicategories as a *1-category*: our morphisms between quasicategories are just morphisms of the 1-category of simplicial sets. At the same time, we have some hint of higher structure of morphisms in the form of $E[1]$-homotopies and $E[1]$-homotopy equivalences. Indeed, we would expect that the collection of all $(\infty, 1)$-categories somehow has the structure of an $(\infty, 1)$- (or maybe $(\infty, 2)$-) category.

In fact, it happens quite that we stumble across a higher category in the guise of a 1-category in which we have some morphisms we want to think of as "equivalences". Two examples we have seen are the 1-categories of Kan complexes and quasicategories with the $E[1]$-homotopy equivalences. Another example we are looking forward to is the category of contexts of a democratic model of type theory, with the equivalences as usually defined by homotopy type theorists. To some extent, the concept of $(\infty, 1)$-category was invented in order to study situations like these.

Even if we are only interested in the $(\infty, 1)$-categorical structure lurking in these 1-categorical presentations, it also can be useful to have strict avatars for higher structure. With quasicategories, for instance, we conveniently have strictly associative and unital composition of functors between quasicategories given by composition in the 1-category of simplicial sets. With this as one motivation, there is a whole zoo of different kinds of 1-categorical "presentation" that connect different higher-categorical structures to 1-categorical avatars. For example, a *Quillen model category* is a presentation of an $(\infty, 1)$-category with all $(\infty, 1)$-limits and -colimits by an 1-category with all 1-limits and -colimits, with additional structure that determines when a 1-(co)limit can be understood as the avatar of a higher (co)limit.

In this course, we are interested in particular in $(\infty, 1)$-categories with *finite limits*, and a way to present such higher categories by 1-categories might be a helpful intermediary in our quest to relate them to models of type theory. For this purpose, we now define *fibration categories*. The definition we give below is used by Szumiło [Szu17], who proves a precise correspondence between such structures and quasicategories with finite limits. There are a number of closely related notions in the literature with various overlapping names, going back to Ken Brown's *categories with fibrant objects* [Bro74]. Rădulescu-Banu's notes [Răd09] are a good reference for the history of and relationship beetween these notions, as well as for the basic theory of (co)fibration categories.

Definition 2.1. A *fibration category* $\mathcal{C} = (\mathcal{C}_0, \mathcal{W}, \mathcal{F})$ is a category $\mathcal{C}_0$ together with two classes of maps $\mathcal{W}$ (the *weak equivalences*, written $\xrightarrow{\sim}$) and $\mathcal{F}$ (the *fibrations*, written $\twoheadrightarrow$), containing all identities and closed under composition, such that

(F0) Weak equivalences satisfy the 2-out-of-6 property.

(F1) Every isomorphism is a weak equivalence and fibration.

(F2) $\mathcal{C}_0$ has a terminal object.

(F3) For every $X \in \mathcal{C}_0$, the unique map $X \to 1$ is a fibration.

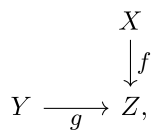
(F4) For any fibration $p\colon Y \twoheadrightarrow X$ and map $f\colon X' \to X$, there exists a pullback square

$$\begin{array}{ccc} Y' & \overset{f'}{\dashrightarrow} & Y \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X' & \xrightarrow[f]{} & X. \end{array}$$

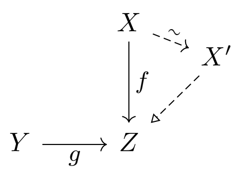
If p is also a weak equivalence, then so is p'.

(F5) Every morphism factors as a weak equivalence followed by a fibration.

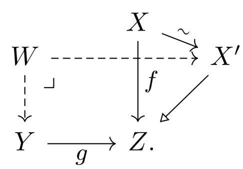
Roughly, the terminal object and pullbacks in a fibration category are supposed to represent a terminal object and pullbacks in some "underlying" $(\infty, 1)$-category. The class of *fibrations* is the key structure that lets us relate the 1-categorical and higher structure: not all 1-categorical pullbacks in a fibration category are meaningful, but pullbacks along fibrations do represent $(\infty, 1)$-categorical pullbacks; ensuring this is the role of the axiom (F4). The axiom (F5) tells us that up to weak equivalence, *every* map is a fibration. In a fibration category, if we want to find (an avatar of) the higher-categorical pullback of an arbitrary cospan



we can first replace one map (say, f) by a fibration using (F5):



and then take the pullback given to us by (F4):



Note the similarity with the way we constructed finite limits in the category of contexts of a type theory with Σ and extensional identity types in the first lecture! In that case, we had "special" pullbacks along context projections, which could be used to construct general pullbacks but also satisfied extra strict equations amongst themselves; something similar is happening here.

In the next lecture, we will see some concrete examples, in particular a fibration category of quasicategories.

References

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