Lecture 4: More fibration categories

Homotopical semantics of type theory

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1 Examples of fibration categories

At the end the previous lecture, we saw a definition of *fibration categories*, which are intended to be somewhat strict presentations of $(\infty, 1)$-categories with finite limits. In this lecture we'll explore fibration categories in more detail, and we'll see in a more precise sense what we mean by the $(\infty, 1)$-category presented by a fibration category. For reference, let's recall the definition:

Definition 1.1. A *fibration category* $\mathcal{C} = (\mathcal{C}_0, \mathcal{W}, \mathcal{F})$ is a category $\mathcal{C}_0$ together with two classes of maps $\mathcal{W}$ (the *weak equivalences*, written $\xrightarrow{\sim}$) and $\mathcal{F}$ (the *fibrations*, written $\twoheadrightarrow$), containing all identities and closed under composition, such that

(F0) Weak equivalences satisfy the 2-out-of-6 property.

(F1) Every isomorphism is a weak equivalence and fibration.

(F2) $\mathcal{C}_0$ has a terminal object.

(F3) For every $X \in \mathcal{C}_0$, the unique map $X \to 1$ is a fibration.

(F4) For any fibration $p\colon Y \twoheadrightarrow X$ and map $f\colon X' \to X$, there exists a pullback square

$$\begin{array}{ccc} Y' & \overset{f'}{\dashrightarrow} & Y \\ {\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X' & \underset{f}{\longrightarrow} & X. \end{array} \tag{1.1}$$

If p is also a weak equivalence, then so is p'.

(F5) Every morphism factors as a weak equivalence followed by a fibration.

We think of the objects and morphisms of a quasicategory as—not necessarily unique—representatives for objects and morphisms of a higher category. The strict limits offered by (F2) and (F4) are likewise meant to represent higher-categorical limits. Note that a general 1-categorical pullback square

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X' & \longrightarrow & X \end{array}$$

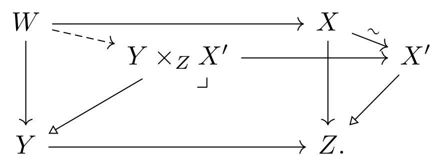
in a fibration category may just be an "artifact of the presentation", i.e., not represent anything on the higher categorical level. When one of the maps is a fibration, however, we are allowed to think of it as a genuine higher-categorical pullback. The other important ingredient if (F5), which says more or less that "every morphism is a fibration up to weak equivalence". Using (F5), we'll be able to define what it means for an arbitrary commutative square to represent a higher-categorical pullback, whether or not it is a 1-categorical pullback square.

Definition 1.2. An object X in a fibration category is *homotopy terminal* if the unique map $X \to 1$ is a weak equivalence.

Definition 1.3. A square

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is a *homotopy pullback* if, for some (equivalently, every[1]) factorization $X \xrightarrow{\sim} X' \twoheadrightarrow Z$, the dashed comparison map



to the strict pullback along $X' \twoheadrightarrow Z$ is a weak equivalence.

By definition, the strict terminal object is also homotopy terminal, and any pullback square of the form (1.1) is also a homotopy pullback square.

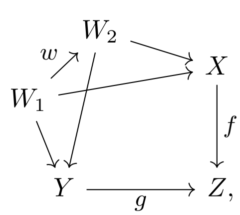
Remark 1.4. If

$$\begin{array}{ccc} W_1 & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \xrightarrow[g]{} & Z \end{array} \qquad \begin{array}{ccc} W_2 & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \xrightarrow[g]{} & Z \end{array}$$

are two homotopy pullbacks of the same span, then we have weak equivalences

$$W_1 \xrightarrow{\sim} Y \times_Z X' \xleftarrow{\sim} W_2$$

to some strict pullback. However, we cannot in general get a weak equivalence $W_1 \xrightarrow{\sim} W_2$ (or $W_2 \xrightarrow{\sim} W_1$). Unlike an isomorphism in a quasicategory, a weak equivalence in a fibration category does not necessarily have an "inverse" on the strict level. If, however, we *do* have a morphism $w \colon W_1 \to W_2$ between the two homotopy pullback cones, as in the diagram



then it follows from 2-out-of-3 that this map w is a weak equivalence.

Remark 1.5. Because $X \to 1$ is a fibration for every X, every strict product $X \times Y$ is also a homotopy product. There are variant definitions of fibration category that do not impose the condition (F4), and for those variants this is not always the case.

We could go on to prove that homotopy pullbacks satisfy other properties we might expect. For example:

[1] This is a basic but not completely trivial exercise.

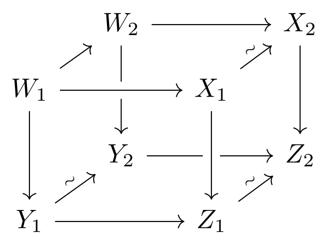
Exercise 1.6 (Homotopy pullback pasting)**.** In a fibration category, given a diagram of the form

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

where the right square is a homotopy pullback, it follows that the left square is a homotopy pullback if and only if the outer rectangle is a homotopy pullback.

An an example, here is a difficult exercise (a corollary of [Răd09, Lemma 1.4.1(2)]) which expresses that the notion of homotopy pullback is in some sense "homotopy-invariant":

Proposition 1.7. In a fibration category, given a commutative cube



where the front and back faces are homotopy pullback squares and the maps $X_1 \to X_2$, $Y_1 \to Y_2$, and $Z_1 \to Z_2$ are weak equivalences, it follows that the map $W_1 \to W_2$ between the homotopy pullback objects is also a weak equivalence.

Remark 1.8. It is also possible to define what it would mean for a non-commutative square in a fibration category to "commute up to homotopy" and generalize our definition of homotopy pullback to squares of this kind.

1.1 The fibration categories of Kan complexes and quasicategories

Conveniently, we already have discussed some categories which we can use as examples of fibration categories: the Kan complexes and the quasicategories.

We previously introduced the notions of $E[1]$-homotopy and $E[1]$-homotopy equivalence in these categories. To recall, $E[1] = \{0 \xrightarrow{\cong} 1\} \in \mathbf{Cat}$ was the walking isomorphism category and $NE[1] \in \mathbf{sSet}$ was its simplicial nerve (i.e., the incarnation of this 1-category as a quasicategory). Given functors of quasicategories $F, G \colon X \to Y$, we defined an *$E[1]$-homotopy $H \colon F \sim G$ between them* to be a functor $H \colon X \times NE[1] \to Y$ such that $H(-,0) = F$ and $H(-,1) = G$. Finally, we said that a functor of quasicategories $F \colon X \to Y$ is an *$E[1]$-homotopy equivalence* when there is a functor $G \colon Y \to X$ with homotopies $H \colon GF \sim \mathrm{Id}_X$ and $K \colon FG \sim \mathrm{Id}_Y$.

For the fibration categories of both Kan complexes and quasicategories, we will define the weak equivalences to be the $E[1]$-homotopy equivalences. It remains to define the fibrations (and then check all the axioms). In both cases, the fibrations will be the maps satisfying a *lifting property*, so let's start with that general concept.

Definition 1.9. Let $f \colon A \to B$ and $p \colon Y \to X$ be morphisms in a category $\mathcal{C}$. We say that *p right lifts against f* if for every commutative square of the form

$$\begin{array}{ccc} A & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & X \end{array}$$

there is a dashed arrow making the diagram

$$\begin{array}{ccc} A & \longrightarrow & Y \\ {\scriptstyle f}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & X \end{array}$$

commute.

In the relevant examples, we want to think of a lifting problem

$$\begin{CD} A @>{y}>> Y \\ @V{f}VV @VV{p}V \\ B @>>{x}> X \end{CD}$$

as follows: A and B are some kind of shapes, and the maps $y\colon A \to Y$ and $: B \to X$ describe A- and B-shaped diagrams in Y and X respectively. The fact that the square commutes tells us that the composite $py\colon A \to X$—the projection of our A-shaped diagram down to X—has an extension to a B-shaped diagram $x\colon B \to X$. The lifting property tells us that in this situation, we can also get an extension of y to a B-shaped diagram $B \to Y$ that lives over the x in p.

Here are a few simple closure properties you can check for yourself:

Exercise 1.10. Suppose $p\colon Y \to X$ and $q\colon Z \to Y$ both right lift against a map $f\colon A \to B$. Show that their composite $pq\colon Z \to X$ also right lifts against f.

Exercise 1.11. Suppose we have a pullback square

$$\begin{CD} Y' @>{u}>> Y \\ @V{p'}VV @VV{p}V \\ X' @>>{v}> X \end{CD}$$

Show that if p right lifts against $f\colon A \to B$, then p' also right lifts against f.

As a special case, we can say that an object $Y \in \mathcal{C}$ right lifts against a map $f\colon A \to B$ when the unique map $p\colon Y \to 1$ right lifts against f. Some properties of simplicial sets we have already discussed can be phrased as such lifting properties:

Example 1.12. A simplicial set $Y \in \mathbf{sSet}$ is a quasicategory if and only if the unique map $Y \to 1$ right lifts against every inner horn inclusion $\Lambda^n_k \rightarrowtail \Delta^n$ (with $n \geq 2$ and $0 < k < n$). It is a Kan complex if and only if the unique map $Y \to 1$ right lifts against every horn inclusion $\Lambda^n_k \rightarrowtail \Delta^n$ (with $n \geq 1$ and $0 \leq k \leq n$).

Now let us define our fibrations of Kan complexes.

Definition 1.13. A morphism of simplicial sets $F\colon Y \to X$ is a *Kan fibration* if it right lifts against every (inner or outer) horn inclusion $\Lambda^n_k \rightarrowtail \Delta^n$ (i.e., for $n \geq 1$ and $0 \leq k \leq n$).

We will also make use of the following definition and fact:

Definition 1.14. A morphism of simplicial sets is an *isofibration* if it has the right lifting property against the two points $0, 1\colon \Delta^0 \to NE[1]$.

Proposition 1.15. Every Kan fibration is an isofibration.

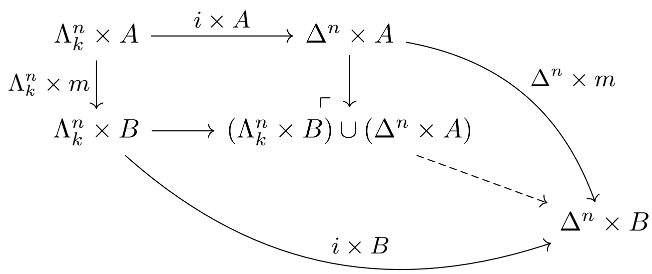
Theorem 1.16. There is a fibration category structure on the category of Kan complexes and functors between them where

(a) the weak equivalences are the $E[1]$-homotopy equivalences, and

(b) the fibrations are the Kan fibrations.

The interesting axioms to check are (F4) (pullbacks) and (F5) (factorization). An important property, now and later on, is the following lemma, which tells us that any Kan fibration actually lifts against a larger class of maps:

Lemma 1.17. Let $m\colon A \rightarrowtail B$ be a monomorphism of simplicial sets and let $i\colon \Lambda^n_k \rightarrowtail \Delta^n$ be a horn inclusion (where $n \geq 1$ and $0 \leq k \leq n$). If $p\colon Y \twoheadrightarrow X$ is a Kan fibration, then it right lifts against the dashed map



where by $(B \times \Lambda^n_k) \cup (A \times \Delta^n)$ we mean the pushout object $(B \times \Lambda^n_k) \sqcup_{A \times \Lambda^n_k} (A \times \Delta^n)$.

The proof of this lemma is not so easy. (For those of you interested in constructivism, I also note that it is non-constructive in full generality, those it is constructive for a well-circumscribed class of monomorphisms $A \rightarrowtail B$.) A few important consequences of the above lemma are the following:

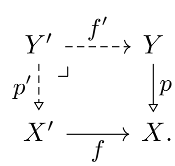
Corollary 1.18. A Kan fibration right lifts against the inclusions $i \times A\colon \Lambda^n_k \times A \rightarrowtail \Delta^n \times A$ for horn inclusion $i\colon \Lambda^n_k \rightarrowtail \Delta^n$ and every $A \in \mathbf{sSet}$.

Corollary 1.19. If $m\colon A \rightarrowtail B$ is a monomorphism and Z is a Kan complex, then $Z^m\colon Z^B \to Z^A$ is a Kan fibration.

Corollary 1.20. If X is a simplicial set and Y is a Kan complex, then every map $X \times \Delta^1 \to Y$ extends to an $E[1]$-homotopy $X \times NE[1] \to Y$.

The existence of pullbacks along fibrations, which is the first half of (F4), is easy to check:

Proposition 1.21. For any Kan complexes Y, X, X', Kan fibration $p\colon Y \twoheadrightarrow X$, and morphism $f\colon X' \to X$, there exists a pullback square



where Y' is also a Kan complex.

Proof. Presheaf categories have all limits, so a pullback exists in simplicial sets. We just have to check that Y' is also a Kan complex and that p' is a fibration. The second one of these follows from Exercise 1.11. To see that Y' is a Kan complex, recall that this is true if and only if $Y' \to 1$ is a Kan fibration. We know that $p'\colon Y' \to X'$ is a Kan fibration, and $X' \to 1$ is a Kan fibration by assumptions, so this follows from Exercise 1.10. □

Remark 1.22. To get some intuition for how Kan fibrations serve our purposes, it can be helpful to unpack the proof that Y' is a Kan complex above. Suppose that we have a horn $h\colon \Lambda^n_k \to Y'$ that we want to fill. By the definition of the strict pullback, it would suffice to fill the images of this horn in X' and Y, $p'h\colon \Lambda^n_k \to X'$ and $f'h\colon \Lambda^n_k \to Y$, in a compatible way. If we only know that X' and Y are Kan complexes, we can indeed find fillers, but they might not project to the same simplex in X. This is where the fibration p comes in. We start by finding a filler $j_0\colon \Delta^n \to X'$ for $p'h\colon \Lambda^n_k \to X'$. Then we can find a filler for $f'h\colon \Lambda^n_k \to Y$ *that matches the image of* f_0 *in* X:

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{f'h} & Y \\ \downarrow & \overset{j_1}{\nearrow} & \downarrow p \\ \Delta^n & \xrightarrow[fj_0]{} & X. \end{array}$$

Now we have enough to build a filler in the pullback:

$$\begin{array}{ccccc} \Delta^n & & \xrightarrow{\quad j_1 \quad} & & \\ & \searrow & & & \\ & & Y' & \xrightarrow{f'} & Y \\ {\scriptstyle j_0}\downarrow & & p'\downarrow & \lrcorner & \downarrow p \\ & \longrightarrow & X' & \xrightarrow[f]{} & X. \end{array}$$

A similar exercise: check that given points $x'\colon \Delta^0 \to X'$ and $y\colon \Delta^0 \to Y$ with an edge $e\colon f(x') \to p(y)$, you can construct a point in y'. This is an indication that Y' is an "object of pairs that are mapped to the same (up to a path) point in X".

The second half of (F4) requires us to check that the pullback of a trivial fibration is a fibration. The key idea is that if a fibration is an $E[1]$-homotopy equivalence, then we can find particularly strict witnesses to that fact:

Definition 1.23. A morphism $f\colon X \to Y$ of simplicial sets is a *deformation retraction* when there is a map $s\colon Y \to X$ such that $ps = \mathrm{id}_X$ and an $E[1]$-homotopy $h\colon sp \sim \mathrm{id}_Y$ such that $ph\colon p \sim p$ is the identity homotopy, i.e., such that the diagram

$$\begin{array}{ccc} Y \times NE[1] & \xrightarrow{h} & Y \\ {\scriptstyle p \times NE[1]}\downarrow & & \downarrow p \\ X \times NE[1] & \xrightarrow[\pi]{} & X, \end{array}$$

commutes.

Any deformation retraction is an $E[1]$-homotopy equivalence, with one homotopy trivial.

Exercise 1.24. Let $p\colon Y \twoheadrightarrow X$ be a Kan fibration. If p is an $E[1]$-homotopy equivalence, then p is a deformation retraction.

Proof hints. By Corollary 1.20, we can work with Δ^1-homotopies instead of $E[1]$-homotopies. Use the fact that p right lifts against $X \times \{0\}\colon X \rightarrowtail X \times \Delta^1$ (Corollary 1.18) to construct your $s\colon X \to Y$ with $ps = \mathrm{id}_X$. Then use right lifting against $(Y \times \{1\}) \cup (X \times \Delta^1) \rightarrowtail Y \times \Delta^1$ (Lemma 1.17) to build the homotopy. □

Exercise 1.25. If p is a deformation retraction, then any pullback of p is a deformation retraction.

Putting together the two previous exercises:

Corollary 1.26. If $p\colon Y \twoheadrightarrow X$ is a fibration and an $E[1]$-homotopy equivalence, then any pullback of p is also an $E[1]$-homotopy equivalence.

That leaves us to check (F5), the existence of factorizations.

Definition 1.27. Given a functor $F\colon X \to Y$ between Kan complexes, define its *mapping path space* to be the pullback

$$\begin{array}{ccc} X \times_Y Y^{NE[1]} & \dashrightarrow & Y^{NE[1]} \\ \downarrow & \lrcorner & \downarrow Y^{\{0\}} \\ X & \xrightarrow[F \times Y]{} & Y. \end{array}$$

The points of the mapping path space of $F\colon X \to Y$ are thus points $x \in X$ paired with isomorphisms $e\colon NE[1] \to Y$ such that $e(0) = F(x)$.

The *mapping path space factorization* of a functor $F\colon X \to Y$ between quasicategories is the factorization

$$X \longrightarrow X \times_Y Y^{NE[1]} \longrightarrow Y$$

where the first map sends a point x to the pair of x with the reflexive isomorphism at $F(x)$, while the second map takes a pair $\langle x, e\rangle$ and projects $e(1)$. (Exercise: describe this two maps in more categorical terms).

Exercise 1.28. Show that the mapping path factorization factors a map between Kan complexes as an $E[1]$-homotopy equivalence followed by a fibration.

Hint. To show that the second map is a fibration, Exercise 1.11 and Corollary 1.19 will be of use. □

That takes care of Kan complexes. We proceed in essentially the same way for quasicategories, but the definition of fibration has an extra wrinkle.

Definition 1.29. A morphism of simplicial sets $F\colon Y \to X$ is an *inner fibration* if it right lifts against every inner horn inclusion $\Lambda^n_k \rightarrowtail \Delta^n$ (i.e., for $n \geq 2$ and $0 < k < n$), an *isofibration* if it has the right lifting property against the two points $0, 1\colon \Delta^0 \to NE[1]$, and an *inner isofibration* if it has both these properties.

Any Kan fibration is automatically an isofibration, which explains why this second condition doesn't come up in the case of Kan complexes.

Theorem 1.30. There is a fibration category structure on the category of quasicategories and functors between them where

(a) the weak equivalences are the $E[1]$-homotopy equivalences, and

(b) the fibrations are the inner isofibrations.

References

[Răd09] Andrei Rădulescu-Banu. *Cofibrations in Homotopy Theory*. 2009. arXiv: math/0610009 [math.AT].