Lecture 5: Localization

Homotopical semantics of type theory

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1 Localization at a class of maps

I keep talking about how fibration categories are supposed to be presentations of quasicategories. It's time to back up a little and explain exactly what I mean by this. My goal for the moment is not to *construct* the quasicategory presented by a fibration category, but just to explain what property what it means for a pre-existing fibration category to present a pre-existing quasicategory.

Most of the structure of a fibration category is actually irrelevant here: only the class of weak equivalences plays a role. Let's say X is a quasicategory, and we want to say what it would mean to give a presentation of X as a 1-category "modulo a class of weak equivalences".

First of all, we should of course have a 1-category $\mathcal{C}$ and a class of maps $\mathcal{W}$ in $\mathcal{C}$. We can regard $\mathcal{C}$ as a quasicategory by taking its simplicial nerve $N\mathcal{C} \in \mathbf{sSet}$, putting $\mathcal{C}$ in the same category as X. If every object and morphism in $\mathcal{C}$ is supposed to represent one in X, then it makes sense to ask for a functor of quasicategories:

$$\gamma \colon N\mathcal{C} \longrightarrow X$$

And if the maps in $\mathcal{W}$ are supposed to represent invertible maps in X, then it is natural to ask that γ sends morphisms in $\mathcal{W}$ (which can also be seen as 1-cells in $N\mathcal{C}$) to invertible 1-cells in X. Finally, we want to say that X and the functor $\gamma \colon N\mathcal{C} \to X$ satisfy a *universal property* with respect to $(\mathcal{C}, \mathcal{W})$: that X is the best quasicategorical approximation to $\mathcal{C}$ in which the maps in $\mathcal{W}$ are invertible.

Before giving an actual definition, let's first generalize a little bit. We're essentially talking about what it means for a quasicategory to be a "quotient" of a 1-category obtained by forcing some maps to be invertible. From that perspective, it's not important that the category we're quotienting is a 1-category: it could also be a quasicategory. So now we'll define what it means for a functor of quasicategories $\gamma \colon Y \to X$ to exhibit X as a "quotient" of Y by a class of 1-cells $\mathcal{W} \subseteq Y([1])$.

Remark 1.1. Let X be a quasicategory and let $P \subseteq X([0])$ be a subset of its 0-cells. Then the sub-simplicial-set $X_P \in \mathbf{sSet}$ defined by

$$X_P([n]) \coloneqq \{x \in X([n]) \mid \forall e \colon [0] \to [n], xe \in P\}$$

is also a quasicategory, the *full sub-quasicategory spanned by P*.

Definition 1.2. Let Y be a quasicategory and let $\mathcal{W} \subseteq Y([1])$ be a subset of the 1-cells of X. For another quasicategory X, write $X^{(Y,\mathcal{W})}$ for the full sub-quasicategory of the exponential $X^Y \in \mathbf{sSet}$ spanned by those functors $F \colon Y \to X$ such that $F_{[1]}(f)$ is invertible for every $f \in \mathcal{W}$.

Definition 1.3. Let $Y \in \mathbf{sSet}$ be a quasicategory and let $\mathcal{W} \subseteq Y([1])$ be a subset of the 1-cells of Y. A *localization* of Y at $\mathcal{W}$ is a quasicategory $L_{\mathcal{W}}Y$ with a functor

$$\gamma \colon Y \longrightarrow L_{\mathcal{W}}Y$$

such that

- γ sends morphisms in $\mathcal{W}$ to invertible maps;
- for every quasicategory Z, precomposition $\gamma^*\colon Z^{L_\mathcal{W} Y} \to Z^{(Y,\mathcal{W})}$ is an $E[1]$-homotopy equivalence of quasicategories.

Now we can say in particular that a fibration category $(\mathcal{C}_0, \mathcal{W}, \mathcal{F})$ presents a quasicategory X when we have some $\gamma\colon N\mathcal{C}_0 \to X$ exhibiting X as the localization of $\mathcal{C}_0$ at $\mathcal{W}$. The key theorem that makes fibration categories useful is that any quasicategory that is a localization of a fibration category has finite (quasicategorical) limits.

Often, though, we don't just want to say that some pre-existing quasicategory is the localization of some fibration category: we start with a fibration category and want to *construct* the localization.

2 Cofibration categories for Kan complexes

On the way to constructing localizations, I'm going to go on a little detour and explain in more detail something I mentioned at the end of the last lecture: *cofibration* categories of ∞-groupoids and $(\infty, 1)$-categories. There is a version of the following story for quasicategories as well as Kan complexes, but let's just go through the case of Kan complexes, and then I'll remark that everything works the same way for quasicategories.

We expect that the $(\infty, 1)$-category of spaces has not just finite limits, but also finite colimits (not to mention larger colimits and limits, exponentials, and more!). We might hope to show this by exhibiting this higher category as a localization of a *cofibration* category, that is, the dual of a fibration category.

When we defined the fibration category of Kan complexes, we used the fact that in certain cases we can take a pullback of a span of Kan complexes and get a Kan complex back. In that way, we could model $(\infty, 1)$-category pullbacks of spaces with 1-categorical pullbacks of Kan complexes. However, the category of Kan complexes has very few strict colimits, since a colimit can easily create new horns without fillers. Therefore, we build a cofibration category structure on the category of *all* simplicial sets.

This makes things a bit harder, since for example $E[1]$-homotopy equivalence is not a well-behaved definition of weak equivalence for arbitrary simplicial sets. We will exploit the following result from the previous lecture:

Corollary 2.1. If $m\colon A \rightarrowtail B$ is a monomorphism and Z is a Kan complex, then $Z^m\colon Z^B \to Z^A$ is a Kan fibration.

As special case of the above (namely where $A = \emptyset$ is the empty simplicial set), if B is any simplicial set and Z is a Kan complex then Z^B is a Kan complex. Now we define:

Definition 2.2. A morphism of simplicial sets $f\colon A \to B$ is a *weak homotopy equivalence* when for every Kan complex $Z \in \mathbf{sSet}$, the map $Z^f\colon Z^B \to Z^A$ is an $E[1]$-homotopy equivalence.

Theorem 2.3. There is a cofibration category structure on **sSet** where

(a) the weak equivalences are the weak homotopy equivalences, and

(b) the cofibrations are the monomorphisms.

I won't prove this theorem, but I'll comment on some aspects of the construction.

Remark 2.4. The dual of condition (F3), the existence of pushouts, is easy to prove with the help of the proof of (F3) for the fibration category of Kan complexes. To take a pushout of a map

$f\colon A \to A'$ along a monomorphism $m\colon A \rightarrowtail B$, we take the pushout in simplicial sets

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m'} \\ B & \xrightarrow[f']{} & B' \end{array}$$

and use the general fact that monomorphisms are closed under pushout in presheaf categories. We also need to know that if m is a weak homotopy equivalence, then so is its pushout. For any Kan complex Z, exponentiating $Z^{(-)}\colon \mathbf{sSet} \to \mathbf{sSet}$ sends colimits to limits, so we have a *pullback* square with everything flipped over:

$$\begin{array}{ccc} Z^{B'} & \xrightarrow{Z^{f'}} & Z^{B} \\ {\scriptstyle Z^{m'}}\big\downarrow & & \big\downarrow{\scriptstyle Z^{m}} \\ Z^{A'} & \xrightarrow[Z^{f}]{} & Z^{A} \end{array}$$

The vertical maps are fibrations by Corollary 2.1. If m is a weak homotopy equivalence, then by definition Z^m is an $E[1]$-homotopy equivalence, and the laws of fibration categories then tell us that $Z^{m'}$ is also an $E[1]$-homotopy equivalence.

Remark 2.5. We can construct factorizations in a similar way. Given a map $f\colon A \to B$, we define its *mapping cylinder* as the pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle A\times\{1\}}\big\downarrow & & \big\downarrow \\ A \times NE[1] & \longrightarrow & C(f) \end{array}$$

The mapping cylinder fits into a factorization $A \to C(f) \to B$ of f. The first map $A \rightarrowtail C(f)$ is easily seen to be a monomorphism. We can see that $C(f) \to B$ is a weak homotopy equivalence by observing that $Z^{(-)}$ sends the mapping cylinder factorization to the mapping *path* factorization of the map between Kan complexes Z^f, then using results about this factorization from the last lecture.

We expanded our category from Kan complexes to all simplicial sets in order to get enough pushouts to build a cofibration category structure. If the idea is that this cofibration category represents a higher category of ∞-groupoids, however, then we should expect that every $A \in \mathbf{sSet}$ is weak homotopy equivalent to some Kan complex. This is indeed the case:

Proposition 2.6. For every simplicial set $A \in \mathbf{sSet}$, there is a weak homotopy equivalence $A \xrightarrow{\sim} X$ where $X \in \mathbf{sSet}$ is a Kan complex. In fact, $A \xrightarrow{\sim} X$ can be chosen to be a monomorphism.

There are multiple ways to construct this *Kan fibrant replacement* of a simplicial set. One is the *small object argument*; another is *Kan's* Ex^∞ *functor*. The small object argument constructs X and its map from A by repeatedly gluing on fillers for all horns until eventually the result is a Kan complex.

Finally, there is a version of this whole story for quasicategories, where we probe with $Z^{(-)}$ for every quasicategory Z instead of just Kan complexes. We'll call the resulting notion of weak equivalence *weak categorical equivalence* to disambiguate from weak homotopy equivalence, as these don't agree outside the subcategory of Kan complexes.

Theorem 2.7. There is a cofibration category structure on $\mathbf{sSet}$ where

(a) the weak equivalences are the weak categorical equivalences, and

(b) the cofibrations are the monomorphisms.

Remark 2.8. I should remark that no one really sets up the homotopy theory of Kan complexes in the way that we have. Usually one defines a *Quillen model category* structure on the category of simplicial sets. This is another kind of presentation of an $(\infty, 1)$-category which covers both finite limits and colimits together. A Quillen model structure on a category consists of classes of weak equivalences, cofibrations, and fibrations, but has a bit more information about how these interact with each other than you would get by naively merging the definitions of cofibration and fibration category. Still, the constructions we've seen would play a role in defining a model category as well.

3 Constructing localizations

Using the tools we just talked about, we can now construct arbitrary localizations. I follow a presentation in Cisinski [Cis19, Proposition 7.1.3].

Definition 3.1. The localization of a quasicategory Y at a subset of its 1-cells $\mathcal{W} \subseteq Y([1])$ exists.

Sketch. We can assume without loss of generality that $\mathcal{W}$ includes all degenerate 1-cells.

Write $Y_{\mathcal{W}}$ for the simplicial set whose 0-cells are the 0-cells of Y, whose 1-cells are the maps in $\mathcal{W}$, and whose higher cells are just the degeneracies of these 0- and 1-cells. By Proposition 2.6, there is a Kan complex $Y_{\mathcal{W}}^{\mathrm{Kan}}$ and a weak homotopy equivalence of Kan complexes $Y_{\mathcal{W}} \rightarrowtail Y_{\mathcal{W}}^{\mathrm{Kan}}$.

We first take a (homotopy) pushout of simplicial sets:

$$\begin{array}{ccc} Y_{\mathcal{W}} & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ Y_{\mathcal{W}}^{\mathrm{Kan}} & \dashrightarrow & Y'. \end{array}$$

Then, applying the version of Proposition 2.6 for quasicategories gives us a weak categorical equivalence $Y' \overset{\sim}{\rightarrowtail} L_{\mathcal{W}}Y$ where $L_{\mathcal{W}}Y$ is a quasicategory.

As the notation suggests, the claim is that the composite $\gamma \colon Y \rightarrowtail Y' \rightarrowtail L_{\mathcal{W}}Y$ is a localization of Y at $\mathcal{W}$. Indeed, we can use the definitions of weak homotopy and categorical equivalence to check that $\gamma^* \colon Z^{L_{\mathcal{W}}Y} \to Z^{(Y,\mathcal{W})}$ is an $E[1]$-homotopy equivalence for all quasicategories Z. □

As a special case, we can construct the quasicategory presented by a (co)fibration category, though we do not yet know that the result has finite (co)limits.

References

[Cis19] Denis-Charles Cisinski. *Higher Categories and Homotopical Algebra*. Cambridge University Press, 2019. ISBN: 9781108473200. DOI: 10.1017/9781108588737.