The algebraic small object argument as a saturating operation

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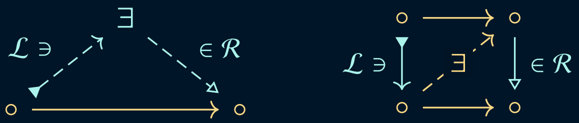
joint work in progress with Christian Sattler

Weak factorization systems

⊗ On a category $\mathscr{E}$, pair $(\mathcal{L}, \mathcal{R})$ of $\mathcal{L}, \mathcal{R} \subseteq \mathrm{Ob}\,\mathscr{E}^{\to}$



Weak factorization systems

- On a category $\mathscr{E}$, pair $(\mathcal{L}, \mathcal{R})$ of $\mathcal{L}, \mathcal{R} \subseteq \mathrm{Ob}\,\mathscr{E}^{\to}$

- Often generated by a set $\mathcal{S} \subseteq \mathrm{Ob}\,\mathscr{E}^{\to}$ of left maps

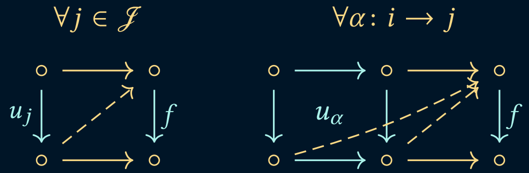
$$\mathcal{R} = \{\text{maps right lifting against all } f \in \mathcal{S}\}$$

- (complemented mono, split epi) in **Set** generated by $\{0 \rightarrowtail 1\}$

- (triv cofib, Kan fib) on $\mathrm{PSh}(\Delta)$ generated by $\{\Lambda^n_k \rightarrowtail \Delta^n \mid k \leq n \in \mathbb{N}\}$

Generation by a category

- f right lifts against $u : \mathscr{J} \to \mathscr{E}^{\to}$ when



- (complemented mono, split epi) in **AbGrp** generated by full subcategory of $\mathscr{E}^{\to}$ containing

$$0 \rightarrowtail \mathbb{Z} \qquad \mathbb{Z} \overset{\Delta}{\rightarrowtail} \mathbb{Z} \times \mathbb{Z}$$

- "uniform fibrations" in, e.g., cubical sets (used in models of HoTT)

Quillen's small object argument

- Build a WFS from a set $\mathcal{S}$ of generators (assuming …)

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$$\begin{array}{c} X \\ \downarrow \\ Y \end{array}$$

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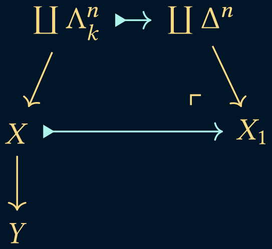

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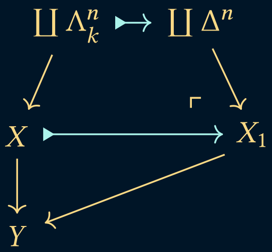

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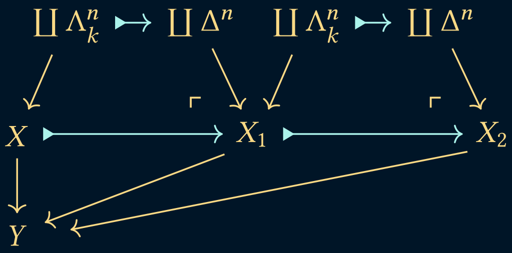

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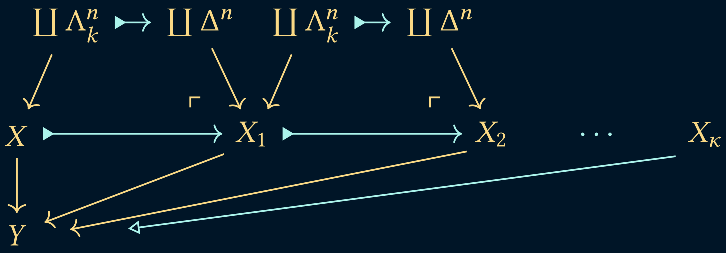

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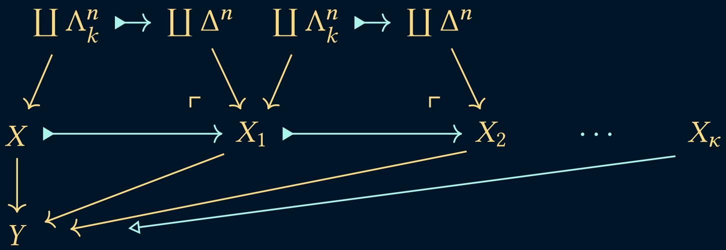

⊗ The left factor is a transfinite composite of cobase changes of coproducts of generating maps

⊗ Any left map is a retract (in $\mathscr{E}^{\to}$) of one of these

Saturation

Def. $\mathscr{A} \subseteq \mathrm{Ob}\,\mathscr{E}^{\rightarrow}$ is saturated when it is closed under coproducts, cobase change, transfinite composition, and retracts.

⊗ When $(\mathcal{L}, \mathcal{R})$ is generated from $\mathcal{S}$ by the SOA, $\mathcal{L}$ is the least saturated class containing $\mathcal{S}$.

Saturation for categories?

- Garner's algebraic small object argument (2008) generates a WFS given $u\colon \mathscr{J} \to \mathscr{E}^{\to}$

- Builds left factors as transfinite composites

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_\kappa$$

... but step maps may not be left maps!

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- : see

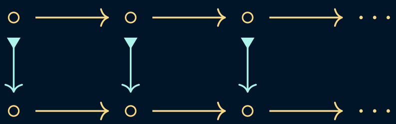
$$\begin{array}{ccccccccc}
X_0 & = & X_0 & = & X_0 & = & \cdots & = & X_0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots & \longrightarrow & X_\kappa
\end{array}$$

  as a colimit in the category **LeftMap** of left-structured maps

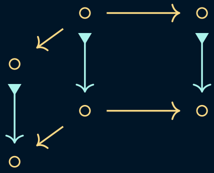
Result

⊗ Left factors are built by colimits of diagrams in **LeftMap**:

(a) sequential colimits



(b) pushouts



(c) colimits of diagrams factoring through $\mathscr{J}$

$$\mathscr{I} \xrightarrow{d} \mathscr{J} \xrightarrow{u} \textbf{LeftMap}$$

⊗ and “vertical” composition $\circ \rightarrowtail \circ \rightarrowtail \circ$

Wrap-up

- Can parlay idea into universal property of the double category of left-structured maps

- Under conditions on $u\colon \mathscr{J} \to \mathscr{E}^{\to}$, form of colimits can be further constrained, *cf.* Athorne's coalgebraic cell complexes (2014)

- Saturation principle for WFS generated by a set is a special case

Thank you!

References

[1] Thomas Athorne. "Coalgebraic Cell Complexes". PhD thesis. University of Sheffield, 2014. URL: etheses.whiterose.ac.uk/6285.

[2] John Bourke and Richard Garner. "Algebraic weak factorisation systems I: Accessible AWFS". In: *Journal of Pure and Applied Algebra* 220.1 (2016), pp. 108–147. DOI: 10.1016/j.jpaa.2015.06.002.

[3] Richard Garner. "Understanding the small object argument". In: *Applied Categorical Structures* 17 (2009), pp. 247–285. DOI: 10.1007/s10485-008-9137-4.

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