Unifying Cubical Models of Univalent Type Theory

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Univalent Type Theory

- Dependent type theory

$$\Gamma \vdash A \text{ type} \qquad \Gamma \vdash M : A$$
$$\Gamma \vdash A = B \text{ type} \qquad \Gamma \vdash M = N : A$$

Univalent Type Theory

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$\Gamma \vdash (a{:}A) \to B$ type	function/implication/∀
$\Gamma \vdash (a{:}A) \times B$ type	product/∃
$\Gamma \vdash \mathbb{N}$ type	inductive types
$\Gamma \vdash \mathrm{Id}_A(M, N)$ type	equality
$\Gamma \vdash \mathcal{U}$ type	universe(s) of types

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e.g. $\cdot \vdash M : (n : \mathbb{N}) \to (m : \mathbb{N}) \times \mathrm{Id}_{\mathbb{N}}(m, n)$

Univalent Type Theory

- **Identity** $\mathrm{Id}_A(a, b)$

$$\frac{a : A}{\mathrm{refl}_A\, a : \mathrm{Id}_A(a, a)} + \frac{a : A \vdash d : P(a, a, \mathrm{refl}_A\, a)}{\mathrm{elim}(a.d, a_0, a_1, t) : P(a_0, a_1, t)}$$

  - Least reflexive relation ($\Rightarrow$ symmetric, transitive, etc.)
  - "Underdetermined"

$$\mathrm{Id}_{(a:A)\to B}(f, g) \overset{?}{\simeq} (a{:}A) \to \mathrm{Id}_B(f\,a, g\,a)$$

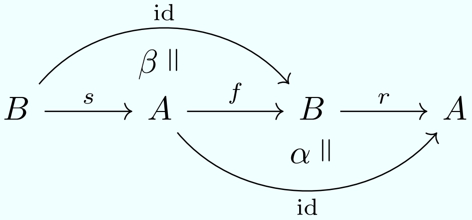
$$\mathrm{Id}_{\mathcal{U}}\ A\ B \simeq\ ?$$

Univalent Type Theory

- Univalence Axiom (Voevodsky)

$$\mathrm{Id}_{\mathcal{U}}(A, B) \simeq (A \simeq B)$$

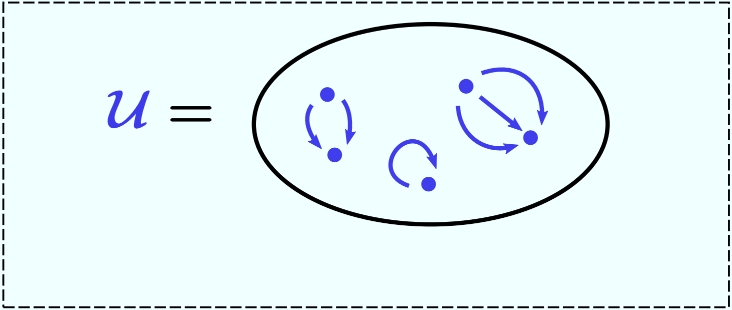
- **Equivalence** $A \simeq B$
  - map $f : A \to B$ with a left and right inverse



$$(f, s, \beta, r, \alpha) : A \simeq B$$

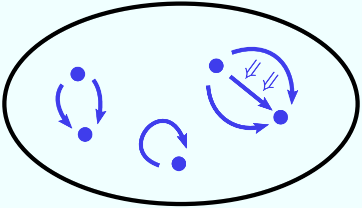
Univalent Type Theory

- Identities are not unique



Univalent Type Theory

- Identities are not unique

$\mathcal{U} =$ 

$\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \mathcal{U}_3 : \cdots$

Univalent Type Theory

- Identities are not unique

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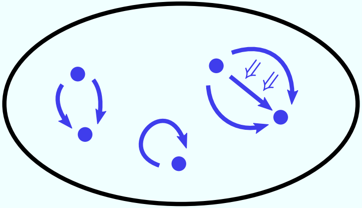


$$\mathcal{U}_0 : \mathcal{U}_1 : \mathcal{U}_2 : \mathcal{U}_3 : \cdots$$

- More: add *higher inductive types*
  - Quotients for proof-relevant identity
  - Language for *synthetic homotopy theory*

Models of Univalent Type Theory

- **Simplicial set model (Kapulkin & Lumsdaine ’12/’18, after Voevodsky)**
  - Classical setting for homotopy theory
  - Essentially non-constructive (Bezem, Coquand, & Parmann ’15)
- **Cubical set model (Bezem, Coquand, & Huber ’13)**
  - First constructive model of univalence
  - Problems with higher inductive types resolved in Cohen, Coquand, Huber, & Mörtberg ’15 and Angiuli, Favonia, & Harper ’18 models

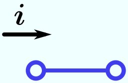
Cubical Set Models

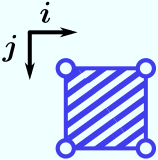
- Interpret contexts as cubical sets
  - family of sets indexed by *interval variable* contexts

$$\Gamma \text{ ctx} \quad \rightsquigarrow \quad [\![\Gamma]\!](i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I}) \text{ for each } n$$

$$\text{“maps } [0,1]^n \to \Gamma\text{”}$$

$[\![\Gamma]\!](\cdot) = \{$  $\}$

$[\![\Gamma]\!](i : \mathbb{I}) = \{$  $\}$

$[\![\Gamma]\!](i : \mathbb{I}, j : \mathbb{I}) = \{$  $\}$

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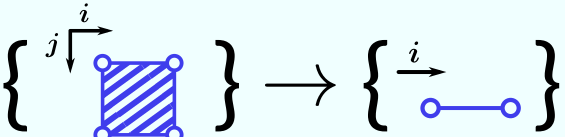
$[\![\Gamma]\!](0/j) :$



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- Interpret types as *fibrations*

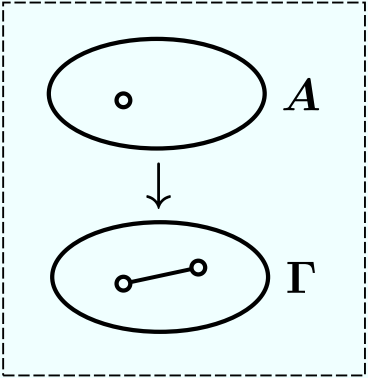
$$\Gamma \vdash A\ \mathrm{type} \rightsquigarrow \text{ family of cubical sets indexed by } [\![\Gamma]\!]$$

$$\text{that “respect paths in } [\![\Gamma]\!]\text{”}$$

Fibrations

- Part 1 (coercion):

"if γ_0 — $\gamma_1 \in [\![\Gamma]\!]$ then $[\![A]\!](\gamma_0) \simeq [\![A]\!](\gamma_1)$"



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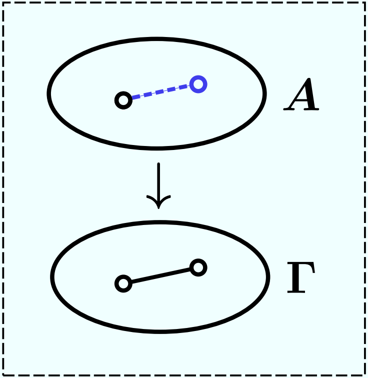

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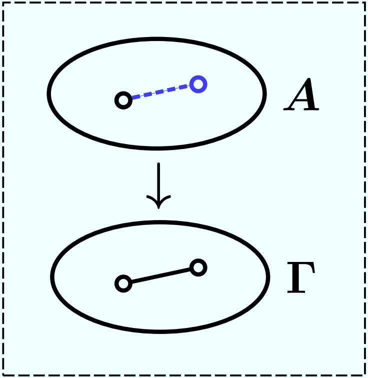

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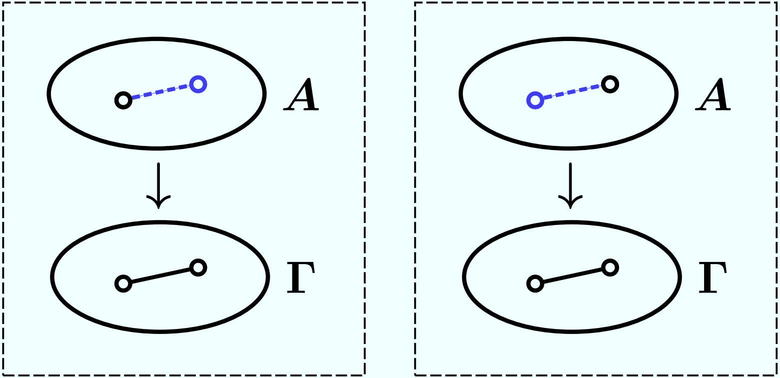

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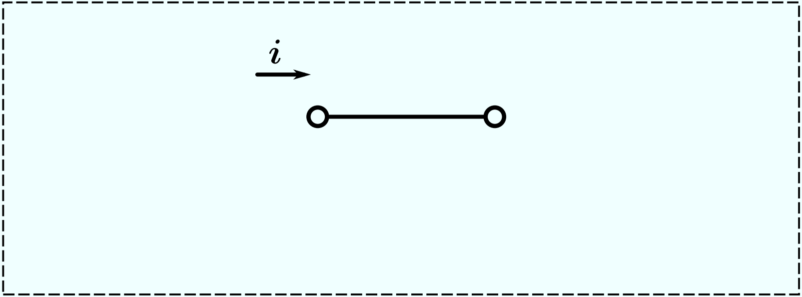

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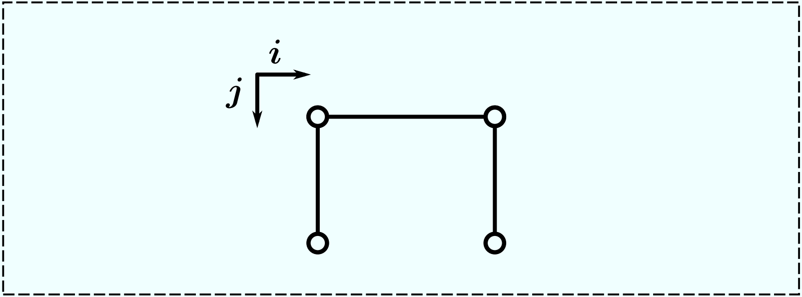

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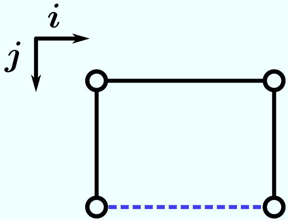

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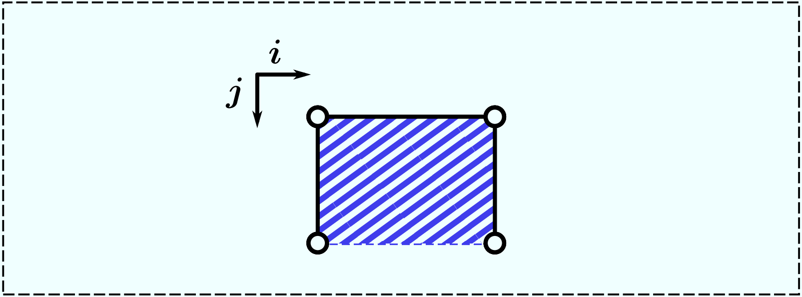

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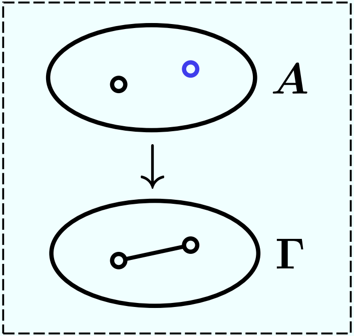
- Part 2 (composition): a cube in A can be adjusted
- A fibration is a family supporting these operations

Two approaches

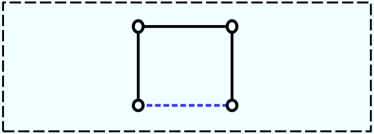
- Cohen, Coquand, Huber, & Mörtberg ’15



coercion $0 \to 1$

$$\neg r, r \sqcup s, r \sqcap s \in \mathbb{I}$$

negation and min/max (connection) operations on $\mathbb{I}$

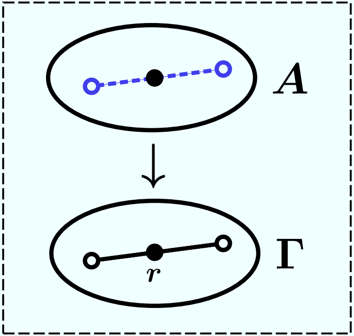


composition adjusts faces

- Result: fibrations closed under type formers

Two approaches

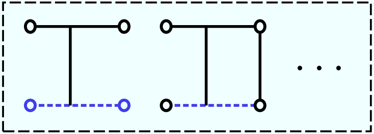
- Angiuli, Favonia, & Harper ’18



coercion $r \to s$

$$x, 0, 1 \in \mathbb{I}$$

only variables and faces in $\mathbb{I}$



composition adjusts any part

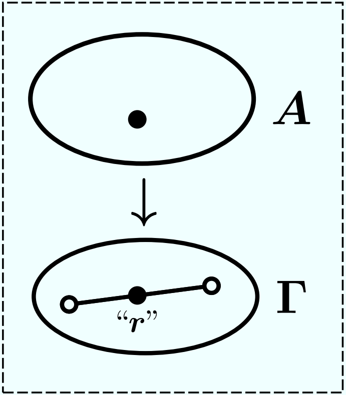
- Result: fibrations closed under type formers

Two approaches

- CCHM
  - Rely on operations on dimensions ($\sqcup, \sqcap, \neg$) to show closure under type formers
  - Does not apply in AFH cubical sets
- AFH
  - Need stronger composition and coercion to show closure under type formers
  - Applies in CCHM cubical sets, but gives inequivalent definition of fibration

Is there a unifying construction that generalizes these?

Unifying construction

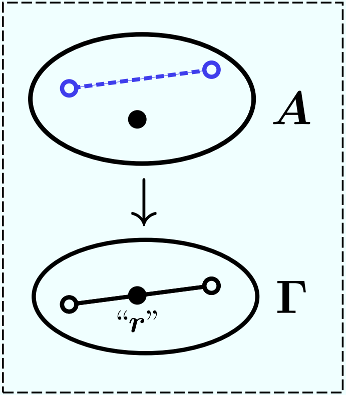


coercion $r \to s$

Q: Where do we use the stronger composition in AFH?

A: Fixing coercion output that doesn't quite agree with input

Unifying construction



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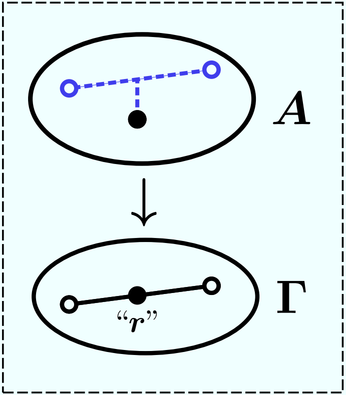

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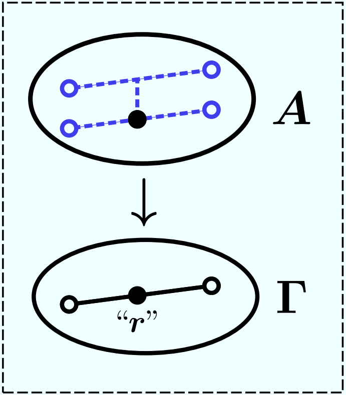

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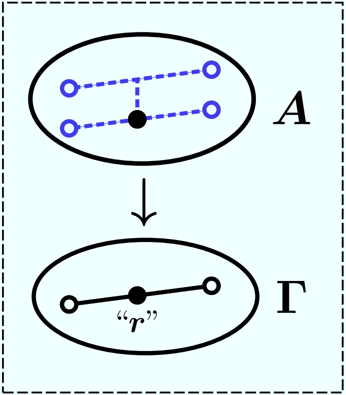

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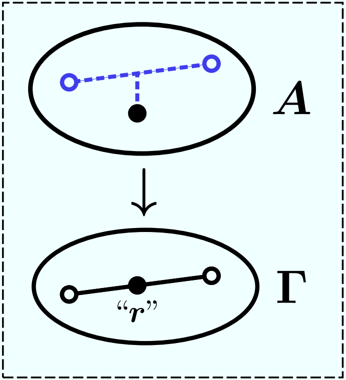
IDEA (CMS):

Weaken the condition on coercion ouput

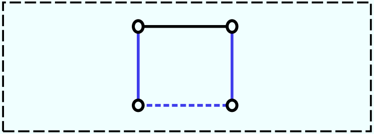
Unifying construction



weak coercion $r \to s$

$$x, 0, 1 \in \mathbb{I}$$

only variables and faces required in $\mathbb{I}$

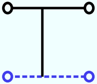


composition adjusts faces
+ arbitrary specified Φ

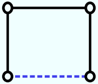
- Fibrations are closed under type formers
- Fibrations participate in a model structure

Unifying construction

- Parameterized by category $\mathcal{C}$ with $\mathbb{I}$ and Φ (+ axioms)

AFH = CMS (cartesian cubical sets , $\mathbb{I}$, )

(strict and weak coercion become equivalent)

CCHM = CMS (De Morgan cubical sets , $\mathbb{I}$, )

($\neg, \sqcup, \sqcap$ + coercion $0 \to 1 \Rightarrow$ weak coercion)

- Also new models, e.g. cartesian w/ only faces in Φ

Unifying construction

- Formulated following Orton & Pitts ’16 (for CCHM),
  Angiuli, Brunerie, Coquand, Favonia, Harper, & Licata ’18 (for AFH)
  - assume $\mathcal{C}$ interprets ordinary type theory
  - describe axioms and construction in internal language
  - enables straightforward formalization (ours in `Agda`)

Unifying construction

- **Model structure**
  - setting for homotopy theory
  - following Sattler '17 (for CCHM)
  - use Swan '18 to translate coercion $r \to s$

$$(C, W, F) \begin{cases} C \text{ (cofibrations): generated by } \Phi \\ W \text{ (weak equivalences): equivalences} \\ F \text{ (fibrations): fibrations} \end{cases}$$

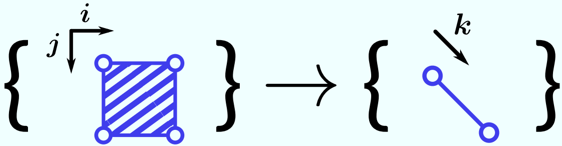
  - Our (C, W, F) has F maximal such that families in F have coercion $0 \to r$

Future work

- **Original cubical model: Bezem, Coquand, & Huber ’13**
  - Substructural: no diagonal maps between cubes

$$(k/i, j) : (k : \mathbb{I}) \to (i : \mathbb{I}, j : \mathbb{I})$$

$[\![\Gamma]\!](k/i,j) :$



  - Definitions of fibration structure for types rely on the *absence* of diagonals

- **How do cubical models relate to other models?**