Internal Parametricity and Cubical Type Theory

Evan Cavallo
& Robert Harper

Carnegie Mellon University

Internal Parametricity

- Bernardy & Moulin.
  A Computational Interpretation of Parametricity. 2012.
  Type Theory in Color. 2013.
- Bernardy, Coquand, & Moulin.
  A Presheaf Model of Parametric Type Theory. 2015.
- Nuyts, Vezzosi, & Devriese.
  Parametric Quantifiers for Dependent Type Theory. 2017.
- C & Harper. [arXiv:1901.00489]
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THIS TALK:

What is internal parametricity, and how does it relate to higher-dimensional type theory?

Parametric polymorphism, intuitively

- Parametric functions are "uniform" in type variables:

$$\lambda a.a \in X \to X$$

$$\lambda a.\lambda b.a \in X \to Y \to X$$

$$\lambda f.\lambda a.f(fa) \in (X \to X) \to X \to X$$

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- Compare with “ad-hoc” polymorphism:

$$\lambda a.\left[\begin{array}{ll} \textbf{true}, & \text{if } X = \textbf{bool} \\ a, & \text{otherwise} \end{array}\right] \in X \to X$$

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- **DEF:** A family of (set-theoretic) functions is **parametric** when it preserves all relations.

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$$\text{for all sets } \boldsymbol{A, B} \text{ and } \boldsymbol{R \subseteq A \times B},$$

$$\boldsymbol{R(a, b)} \text{ implies } \boldsymbol{R(F_A(a), F_B(b))}$$

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- **Abstraction theorem**: the denotation of any term in simply-typed λ-calculus (with ×, bool) is parametric.

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$$F_A(a) = a$$

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$$\exists n \in \mathbb{N}.\ F_A(f) = f^n$$

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- Abstraction theorem extends interpretation to terms

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- univalence via **G / Glue / V types**

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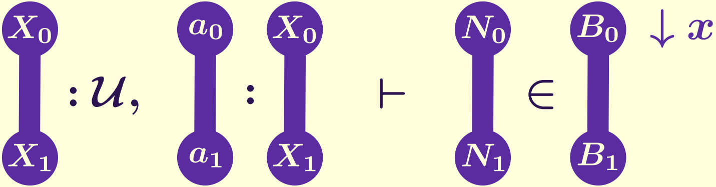

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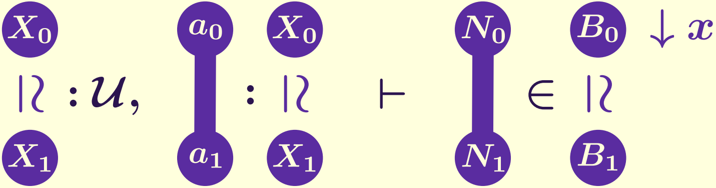

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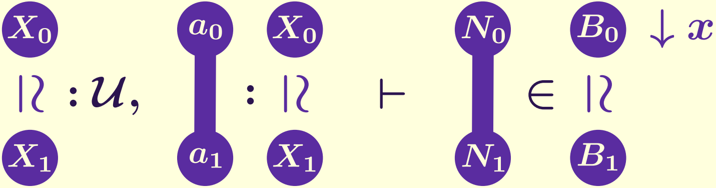

- Can we do the same for relations?

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- Context of **bridge variables** (**colors**)

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Degeneracies: $M \in A\ [\Phi] \Rightarrow M \in A\ [\Phi, \underline{x}]$

Permutations: $M\langle \underline{y}/\underline{x}\rangle$ when $\underline{y} \# M$

Bridge types

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"case analysis for dimension terms"

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BCH G-types for relations

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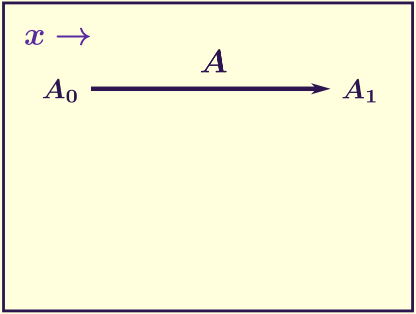

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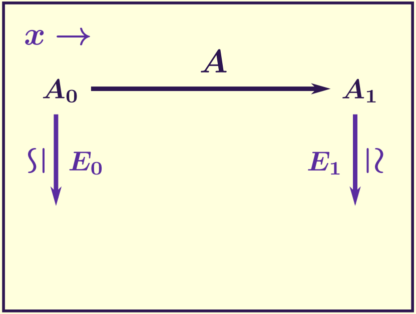

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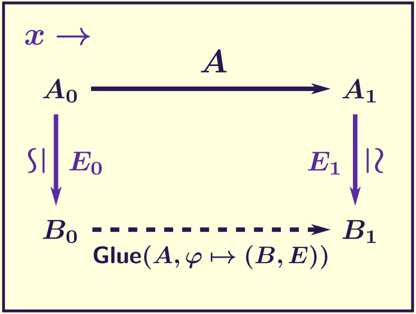

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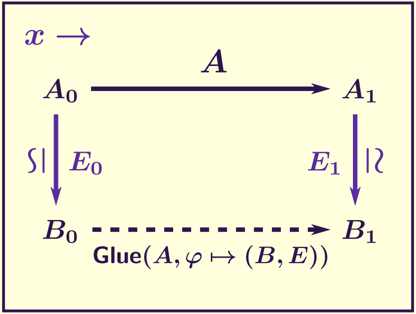

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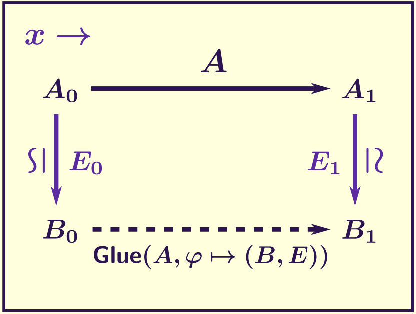

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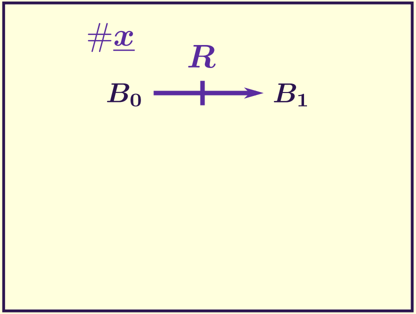

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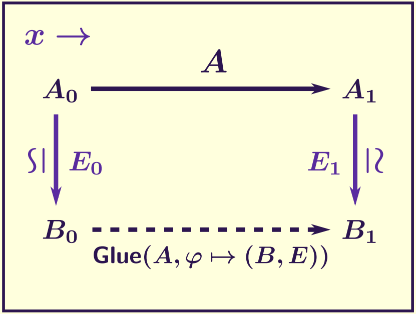

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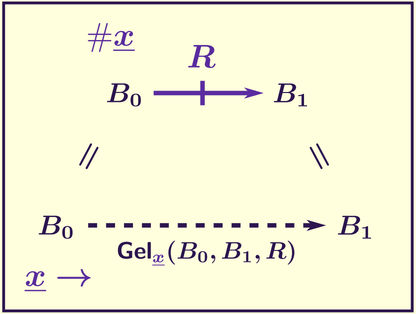

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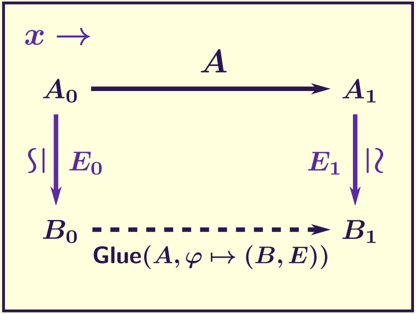

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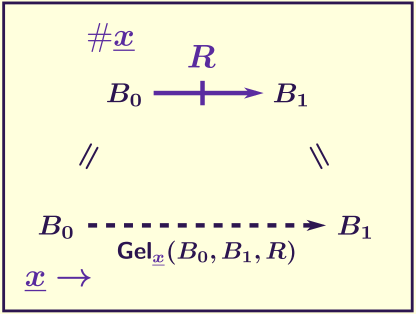

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Bridge$_{_.A}(-,-)$

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- Alternative: use univalence?

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w/ computational semantics

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$$\Longrightarrow\ \mathsf{Bridge}_{\underline{x}.\mathsf{Gel}_{\underline{x}}(A,B,R)}(M, N) \simeq R\langle M, N\rangle$$

(and the other inverse condition)

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\Cap & & \Cap & & \Cap \\
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③ Extract a witness.

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③ Prove this is an equivalence! (**iterated parametricity**)

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(see also: Booij, Escardó, Lumsdaine, & Shulman, *Parametricity, automorphisms of the universe, and excluded middle*)

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(case of **graph lemma** in ordinary parametricity)

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  conjecture: must be constant or identity
  Use to prove pentagon, hexagon, etc