Cubes with one connection and relative elegance

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joint work with
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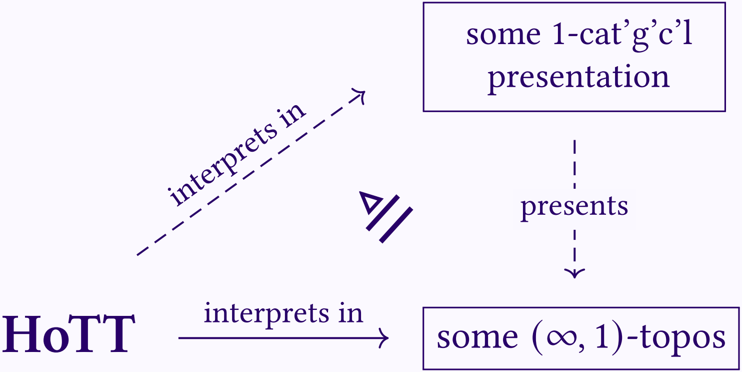
A BIG PICTURE

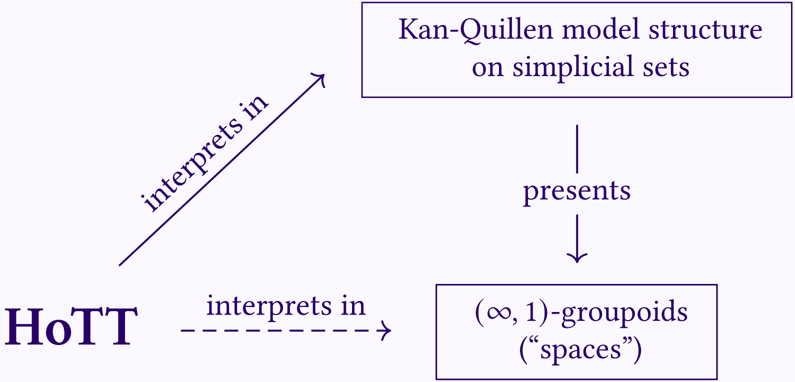


Shulman '19: HoTT interprets in Grothendieck $(\infty,1)$-toposes.

What do we mean by **interpret**?

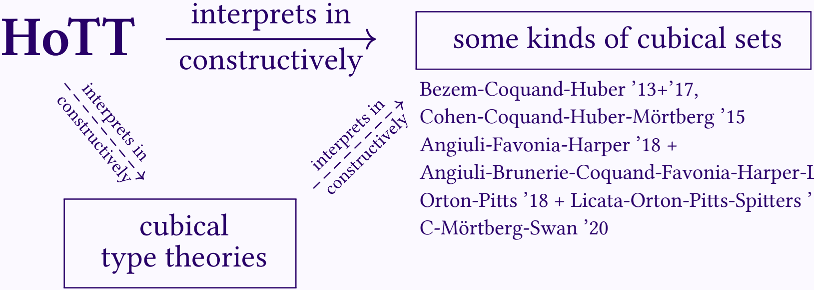
A BIG PICTURE

For example, Voevodsky's simplicial model:



Model structure helps build model of HoTT—but not the same thing

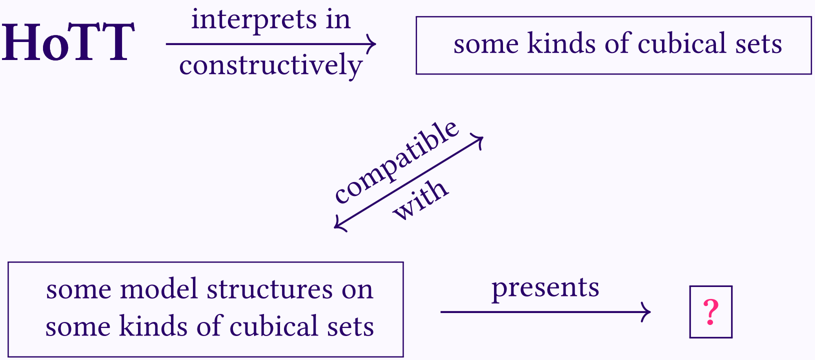
Does HoTT have a constructive interpretation?



Bezem-Coquand-Huber ’13+’17,
Cohen-Coquand-Huber-Mörtberg ’15
Angiuli-Favonia-Harper ’18 +
Angiuli-Brunerie-Coquand-Favonia-Harper-Licata ’21
Orton-Pitts ’18 + Licata-Orton-Pitts-Spitters ’18
C-Mörtberg-Swan ’20

Interpretations of HoTT in a direct sense.

Gambino-Sattler ’17, Sattler ’17, C-Mörtberg-Swan ’20, Awodey:
The cubical interpretations give rise to model structures.



Starter question: do any present $(\infty,1)$-groupoids?

Why want this?

— Present $(\infty,1)$-groupoids **constructively**
(see also Henry '19, Gambino-Sattler-Szumiło '19)

— Interpret **cubical type theories** in $(\infty,1)$-groupoids
(and ideally elsewhere, à la Shulman '19)

CUBE CATEGORIES

Objects are monoidal products of an **interval** $1 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} \mathbb{I}$.

— For cubical type theorists, products are usually **cartesian**:

$\mathbb{I} \xrightarrow{\varepsilon} 1$ degeneracy

$\mathbb{I} \xrightarrow{\Delta} \mathbb{I}^2$ diagonal (except BCH)

$\mathbb{I}^n \xrightarrow{\sigma \in \Sigma_n} \mathbb{I}^n$ symmetries

(unusual from a classical homotopy theory perspective!)

— Extra toppings:

$\mathbb{I}^2 \xrightarrow{\vee} \mathbb{I}$ $\quad \mathbb{I}^2 \xrightarrow{\wedge} \mathbb{I}$ max- and min-connections

$\mathbb{I} \xrightarrow{\neg} \mathbb{I}$ reversal

etc.

CUBE CATEGORIES

Which cube categories lead to model structures presenting spaces?

Ulrik Buchholtz and Christian Sattler investigated in 2018:

Affine (BCH)	$\delta, \varepsilon, \sigma$	✗
Cartesian (AFH+ABCFHL)	$\delta, \varepsilon, \Delta, \sigma$	✗
Dedekind (CCHM)	$\delta, \varepsilon, \Delta, \sigma, \vee, \wedge$	?
De Morgan (CCHM)	$\delta, \varepsilon, \Delta, \sigma, \vee, \wedge, \neg$	✗

In 2019, Awodey-C-Coquand-Riehl-Sattler present a new model:

Cartesian with *equivariant fibrations*	$\delta, \varepsilon, \Delta, \sigma$	✓

Same cube category, stronger lifting condition on types

CUBE CATEGORIES

Our result:

Disjunctive	$\delta, \varepsilon, \Delta, \sigma, \vee$	✓

Compared to equivariant model...

— Easier to describe:
  · in a cartesian cube category with a connection, all fibrations are equivariant

— Proof it presents $(\infty,1)$-groupoids is more direct

Fill out general understanding of cubical models!

DISJUNCTIVE CUBES

No time to give full picture of proof.
(See Sattler '19, Streicher-Weinberger '21 for similar setup.)

What properties of disjunctive cubes matter?

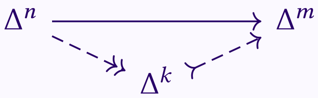
- **bad news:** $\square_\vee$ is not an elegant Reedy category.
- **good news:** it's close to one!

REEDY CATEGORIES

— Thinking of presheaves on **C** as “spaces built from cells shaped like objects of **C**”, useful if:

· objects are stratified by “dimension”

· maps factor into basic “degeneracy”-like and “face”-like maps



Def (~Berger-Moerdijk ’11):
A **(generalized) Reedy category** is a category **R** equipped with a function $|{-}|\colon \operatorname{Ob}\mathbf{R} \to \mathbb{N}$ and orthogonal factorization system $(\mathbf{R}^-, \mathbf{R}^+)$ compatible in the sense that...

— e.g.: simplex category, some cube categories, many more...

- Any presheaf X over a Reedy category $\mathbf{R}$ can be built by iteratively attaching n-cells via colimits

$$\begin{array}{ccccccc} & & \{\text{boundaries}\} & \longrightarrow & \coprod\{\text{2-cells}\} & & \\ & & \downarrow & & \ulcorner\ \downarrow & & \\ X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow \cdots \longrightarrow & \mathrm{colim}_i\, X_i \cong X \end{array}$$

- If $\mathbf{R}$ is **elegant**, then cell maps are monos

  $\Longrightarrow$ If cofibrations=monos, these are also **homotopy colimits**

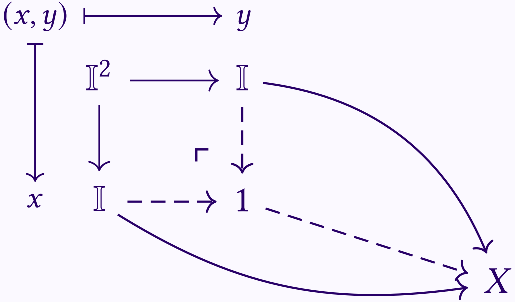
Def (~Berger-Moerdijk '11, ~Bergner-Rezk '13):
A Reedy category $\mathbf{R}$ is **elegant** when

(a) any span of degeneracy maps has a pushout;

(b) any $X \in \mathbf{PSh}(\mathbf{R})$ sends these to pullbacks.

$\Leftrightarrow \mathbf{y} : \mathbf{R} \to \mathbf{PSh}(\mathbf{R})$ preserves them.

— For example with cubes:



· degeneracies with same domain can be "combined"

· a cell is degenerate in two ways iff degenerate in their combination

ELEGANT REEDY CATEGORIES

Want a **Quillen equivalence** with the Kan-Quillen model structure:

$$\mathbf{PSh}(\Delta) \underset{u^*}{\overset{u_!}{\rightleftarrows}} \mathbf{PSh}(\square_\vee) \qquad (u_! \dashv u^*)$$

— These are **left Quillen adjoints**

— So they commute with those colimits—
only need to check they're inverse on "basic cells"?

DISJUNCTIVE CUBES

$$\underset{\text{degeneracy}}{\mathbb{I} \xrightarrow{\varepsilon} 1} \qquad \underset{\text{diagonal}}{\mathbb{I} \xrightarrow{\Delta} \mathbb{I}^2} \qquad \underset{\text{symmetries}}{\mathbb{I}^n \xrightarrow{\sigma \in \Sigma_n} \mathbb{I}^n} \qquad \underset{\text{endpoints}}{1 \overset{\delta_0, \delta_1}{\rightrightarrows} \mathbb{I}} \qquad \underset{\text{max-connections}}{\mathbb{I}^2 \xrightarrow{\vee} \mathbb{I}}$$

— Like other cartesian cube cats, it's a finite product (i.e. Lawvere) theory, the **theory of 01-semilattices**

$$(x \vee y) \vee z = x \vee (y \vee z) \qquad x \vee y = y \vee x$$
$$x \vee x = x \qquad x \vee 0 = x \qquad x \vee 1 = 1$$

Maps $\mathbb{I}^m \longrightarrow \mathbb{I}^n$ are n-tuples of terms in m variables in this language

DISJUNCTIVE CUBES

$$\mathbb{I} \xrightarrow{\varepsilon} 1 \quad \text{degeneracy} \qquad \mathbb{I} \xrightarrow{\Delta} \mathbb{I}^2 \quad \text{diagonal} \qquad \mathbb{I}^n \xrightarrow{\sigma \in \Sigma_n} \mathbb{I}^n \quad \text{symmetries} \qquad 1 \overset{\delta_0, \delta_1}{\rightrightarrows} \mathbb{I} \quad \text{endpoints} \qquad \mathbb{I}^2 \xrightarrow{\vee} \mathbb{I} \quad \text{max-connections}$$

— Also embeds in the **category of semilattices**:

$$\begin{array}{ccc} \square_{\vee} & \longrightarrow & \mathbf{SLat} \\ \mathbb{I}^n & \longmapsto & \{0 < 1\}^n \end{array}$$

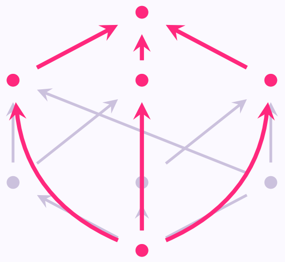
Follows from duality between finite 01-semilattices and finite semilattices:

$$\square_{\vee} \overset{よ}{\hookrightarrow} \mathbf{01SLat}_{\mathrm{fin}}^{\mathrm{op}} \xrightarrow{\simeq} \mathbf{SLat}_{\mathrm{fin}} \hookrightarrow \mathbf{SLat}$$

ARE DISJUNCTIVE CUBES REEDY?

No.

$$\begin{array}{rcl} \mathbb{I}^3 & \longrightarrow & \mathbb{I}^3 \\ (x, y, z) & \longmapsto & (x \vee y, y \vee z, z \vee x) \end{array}$$



Not an iso, but doesn't factor through a lower-degree cube.

(Also doesn't factor in the idempotent completion of $\square_\vee$.)

RELATIVE ELEGANCE

— But know that $\square_\vee$ **embeds** in a Reedy category

$$i \colon \square_\vee \hookrightarrow \mathbf{SLat}_{\mathrm{fin},\top}$$

by general properties of algebraic categories.

— So can borrow cellular decomposition:

$$\begin{array}{ccc}
\mathbf{PSh}(\square_\vee) & \underset{i_*}{\overset{i^*}{\leftrightarrows}} & \mathbf{PSh}(\mathbf{SLat}_{\mathrm{fin},\top}) \\
X & \longmapsto & i_* X \\
\cong & & \cong \\
\mathrm{colims}\,(i^*\mathrm{cells}) & \longleftarrow\!\!\shortmid & \mathrm{colims}\,(\mathrm{cells})
\end{array}$$

RELATIVE ELEGANCE

- To use the decomposition, need cell maps to be monos
- $\mathbf{SLat}_{\mathrm{fin},\top}$ is **not** elegant;
  not all preheaves in $\mathbf{PSh}(\mathbf{SLat}_{\mathrm{fin},\top})$ have good decompositions.
- But it's "elegant relative to $i : \square_\vee \hookrightarrow \mathbf{SLat}_{\mathrm{fin},\top}$":
  presheaves in the image of i_* have good decompositions

> **Def** (C-Sattler):
> A fully faithful $i : \mathrm{C} \to \mathbf{R}$ with $\mathbf{R}$ a Reedy category is **relatively elegant** when
>
> (a) any span of degeneracy maps in $\mathbf{R}$ has a pushout;
>
> (b) $i_* X$ sends these to pullbacks for $X \in \mathbf{PSh}(\mathrm{C})$.
>
> $\Leftrightarrow N_i : \mathbf{R} \to \mathbf{PSh}(\mathrm{C})$ preserves them.

RELATIVE ELEGANCE

Thm (C-Sattler):
If $i : \mathbf{C} \to \mathbf{R}$ is relatively elegant, then any presheaf over $\mathbf{C}$ has a "good" decomposition where the basic cells are $N_i r / N_i G$ for $r \in \mathbf{R}$ and $G \subseteq \mathrm{Aut}_{\mathbf{R}}(r)$.

— Relative elegance of $i : \square_\vee \hookrightarrow \mathbf{SLat}_{\mathrm{fin},\top}$ also follows from general properties of algebraic categories.

— Easy to check basic cells are contractible in this case.

— Have what we need to finish our proof!

In the end:

$$\begin{array}{ccc} & \xrightarrow{\;u_!\;} & \\ & \bot & \\ \mathbf{PSh}(\Delta) & \xleftarrow{\;u^*\;} & \mathbf{PSh}(\square_\vee) \\ & \bot & \\ & \xrightarrow[\;u_*\;]{} & \end{array}$$

- Both of these adjunctions are Quillen equivalences.
- In particular, model structure presents (∞,1)-groupoids!
- Corollary: coincides with the **test model structure** on $\mathbf{PSh}(\square_\vee)$ (compare Streicher-Weinberger ’21)

EXTRAS

— Sad truth: the Dedekind cubes do not embed elegantly in **any** Reedy category.

Some ponderables:

— Comparison with constructive simplicial model structure?

— Other applications for relative elegance or this cube category?

— For cubical-type model structures that don't present spaces,

(a) can we "fix" them? or

(b) can we describe what they **do** present?