Why some cubical models don't present spaces

Evan Cavallo
University of Gothenburg

joint work with
Christian Sattler

Constructivity vs. homotopy theory

"HoTT is a constructive language for homotopy theory"

- *for homotopy theory*:
  - interpret in simplical sets (Kapulkin–Lumsdaine '21)
  - interpret in any ∞-topos (Shulman '19)

- *constructive*:
  - constructive interpretations:
    - in cubical settings (references to come)
    - in simplicial sets? work in progress (Gambino–Henry '19, van den Berg–Faber '22)
  - homotopy canonicity (Kapulkin–Sattler '??, Bocquët '23)

Constructivity vs. homotopy theory

- Classically, have "standard homotopy theory"
  - Topological spaces, simplicial sets, *etc.* are equivalent, present a well-behaved $(\infty, 1)$-*category of spaces*
- Constructive picture more nuanced, still developing (Shulman '21, "The derivator of setoids")
- Starter question:

which *constructive* interpretations
classically present spaces?

Constructivity vs. homotopy theory

which *constructive* interpretations
classically present spaces?

- ⊗ Equivariant fibrations in cartesian cubical sets (Awodey–C–Coquand–Riehl–Sattler '??)
- ⊗ Cartesian cubical sets + one connection (C–Sattler '22)
- ⊗ Constructive simplicial set ~interpretations

Constructivity vs. homotopy theory

this talk:
which constructive interpretations
classically *do not* present spaces?

many cubical interpretations!

- ⊗ ideas sketched in Sattler's 2018 talk
  "Do cubical models of type theory also model homotopy types?"
- ⊗ portion in Coquand's 2018 note
  "Trivial cofibration-fibration factorization with one application"

  @ `groups.google.com/g/homotopytypetheory/c/RQkLWZ_83kQ`
- ⊗ full writeup from Christian and I on the way ☺

Constructivity vs. homotopy theory

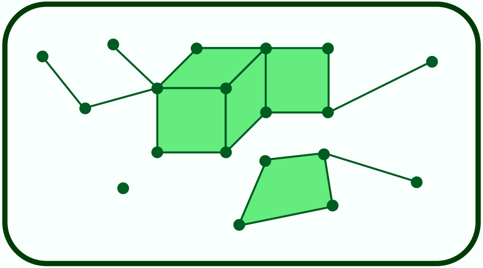
why care?

- ⊗ motivates, *e.g.* “equivariance” fix in cartesian cubes
- ⊗ gives some hint towards characterizing these models?
- ⊗ some general tools for comparing with spaces

Table of contents

Interpreting HoTT in cubical sets

- cubical sets = presheaves on a *cube category* $\square$



A CUBICAL SET

- choice of $\square$ determines structure inherent in a cube
  - does every square have a diagonal?
  - for every edge, is there an edge in the opposite direction?

Kan's cubical sets

- starting point is Daniel Kan '55:
  homotopy theory with the *minimal cube category*
  - objects look like $\mathbf{I} \otimes \cdots \otimes \mathbf{I}$
  - every n-cube has two faces along each axis

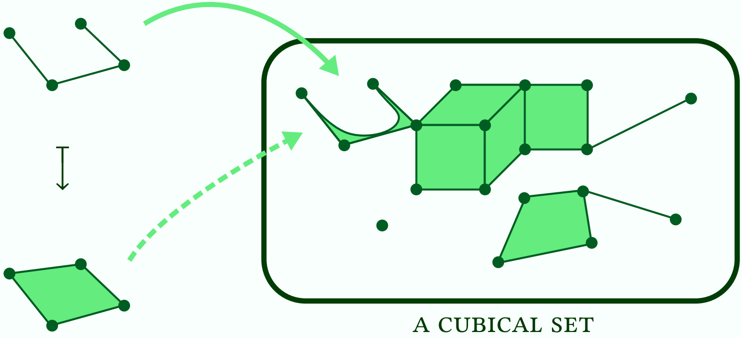
    $$\mathbf{I} \otimes \delta_0 \otimes \mathbf{I} : \mathbf{I} \otimes \mathbf{I} \to \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}$$

    $$\mathbf{I} \otimes \mathbf{I} \otimes \delta_1 : \mathbf{I} \otimes \mathbf{I} \to \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}$$
  - every n-cube can be seen as a degenerate $(n+1)$-cube

    $$\mathbf{I} \otimes \varepsilon : \mathbf{I} \otimes \mathbf{I} \to \mathbf{I}$$
  - and some equations, and that's it.
- the cubical sets that encode spaces are those with *box filling*.

Kan's cubical sets

⊗ *box filling*: every "open box" is filled by a cube



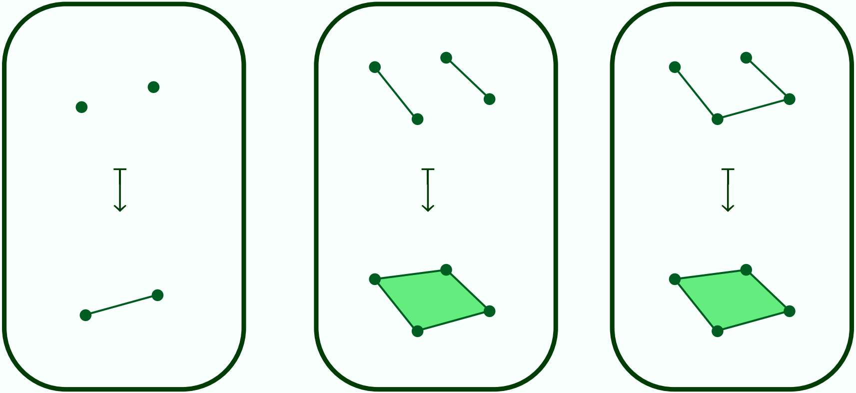
Kan's cubical sets

⊗ how are open boxes formed?

(1) start from the boundary of a cube:

(2) stretch everything in a new direction:

(3) add a "cap" on the top or bottom:



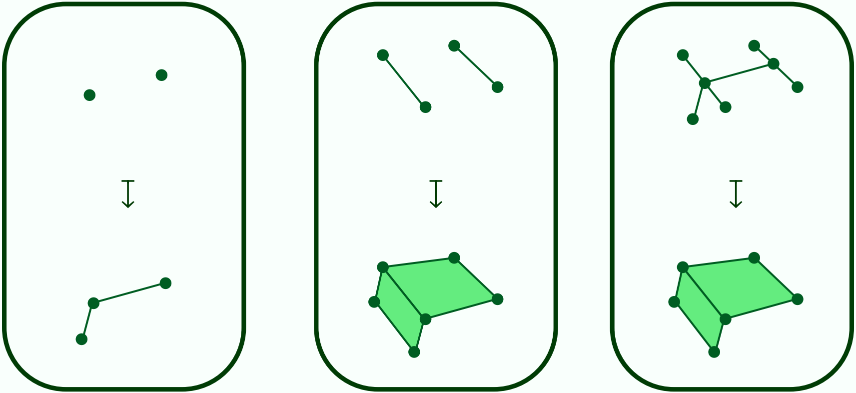
Kan's cubical sets

⊗ how are open boxes formed? — equivalent take 2

(1) start from a subobject of a □-set

(2) stretch everything in a new direction:

(3) add a "cap" at a "generalized point":



Interpreting HoTT in cubical sets

- To model HoTT constructively, want more structured $\square$

- Bezem–Coquand–Huber '13, '19:
  in *affine* cubical sets

$$\mathbf{I} \otimes \mathbf{I} \xrightarrow[\cong]{\sigma} \mathbf{I} \otimes \mathbf{I}$$

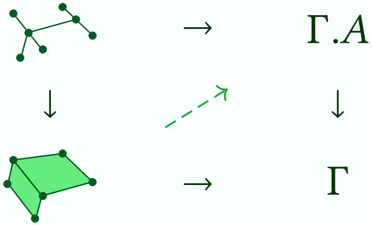
- Cohen–Coquand–Huber–Mörtberg '15:
  in *De Morgan cubical sets*

$$\otimes \text{ is } \times \qquad \mathbf{I} \otimes \mathbf{I} \xrightarrow{\vee,\, \wedge} \mathbf{I} \qquad \left(\mathbf{I} \xrightarrow[\cong]{\neg} \mathbf{I} \right)$$

- Angiuli–Favonia–Harper '18,
  Angiuli–Brunerie–Coquand–Harper–Favonia–Licata '21
  in *cartesian cubical sets*

$$\otimes \text{ is } \times$$

- In all cases, intepret types $\Gamma \vdash A$ by maps with box filling, *i.e.* right lifting against box inclusions



- Is it still homotopically reasonable for these □'s?

Model structures on cubical sets

- Do still get *Quillen model structures* on $\square$-sets (constructively!): (Sattler '17, C–Mörtberg–Swan '20, Awodey '23)

cofibrations ($\rightarrowtail$)
(decidable) monos

fibrations ($\twoheadrightarrow$)
right lift against box inclusions

trivial cofibrations ($\overset{\sim}{\rightarrowtail}$)
left lift against $\twoheadrightarrow$
$\cup\!\!|$
box inclusions

trivial fibrations ($\overset{\sim}{\twoheadrightarrow}$)
right lift against $\rightarrowtail$

weak equivalences ($\xrightarrow{\sim}$) $=$ $\overset{\sim}{\twoheadrightarrow} \circ \overset{\sim}{\rightarrowtail}$

- So, at least well-defined notion of homotopy

Model structures on cubical sets

- Can compare model categories up to *Quillen equivalence*,
  e.g. to the standards on simplicial sets / topological spaces

- Also have *test model structures* on $\square$-sets to compare directly

  - Cisinski '06: any *test category* admits a model structure
    - with $\blacktriangleright\!\!\rightarrow \;=\; \rightarrowtail$
    - Quillen equivalent to simplicial sets

  - Buchholtz-Morehouse '17: our $\square$'s are test categories

  - Doesn't give very explicit def'n of $\twoheadrightarrow$
    —not so easy to compare

What could go wrong?

- Intuition: any space should be an h-colimit of contractible things
- Cubes are made of just one point:

$$1 \xrightarrow[\delta_0]{\sim} \mathbf{I} \xrightarrow[\mathbf{I} \otimes \delta_0]{\sim} \mathbf{I}^2 \xrightarrow[\mathbf{I}^2 \otimes \delta_0]{\sim} \mathbf{I}^3 \xrightarrow{\sim} \cdots$$

- More structure on $\square \implies$ more potentially exotic objects

$$\mathbf{I}^2/\sigma = \operatorname{colim}\left\{ \mathbf{I}^2 \circlearrowleft \sigma \right\}$$

$$\mathbf{I}/\neg = \operatorname{colim}\left\{ \mathbf{I} \circlearrowleft \neg \right\}$$

$$\operatorname{colim}\left\{ \begin{matrix} i,j \mapsto i \wedge j, j \\ \circlearrowright \\ \mathbf{I}^2 \\ \circlearrowleft \\ i,j \mapsto i, i \vee j \end{matrix} \right\}$$

$$\operatorname{Im}\left\{ \mathbf{I}^3 \xrightarrow{i,j,k \mapsto i \wedge j, j \wedge k, i \vee k} \mathbf{I}^3 \right\}$$

What could go wrong?

$$\mathbf{I}^2/\sigma = \operatorname{colim}\left\{\ \mathbf{I}^2 \circlearrowleft \sigma\ \right\} \qquad \mathbf{I}/\neg = \operatorname{colim}\left\{\ \mathbf{I} \circlearrowleft \neg\ \right\}$$

⊗ Topologically, look like they should be contractible

⊗ Sometimes we know they are:

$$\begin{aligned} \mathbf{I}^2/\sigma \times \mathbf{I} &\to \mathbf{I}^2/\sigma \\ (i,j), t &\mapsto (i \vee t, j \vee t) \end{aligned}$$

⊗ What does the test model structure say?

- ⊘ In test cartesian □-sets, $\mathbf{I}^2/\sigma$ is contractible
- ⊘ (Buchholtz) In test De Morgan □-sets, $\mathbf{I}/\neg$ is $K(\mathbb{Z}_2, 1)$
- ⊘ (Sattler) In test affine □-sets, $\mathbf{I}^2/\sigma$ is $\Sigma K(\mathbb{Z}_2, 1)$

Table of contents

Invariants of model categories

- Not enough to show particular realization isn't an equivalence, nor to show that test model structure is different
- Seek property invariant under Quillen equivalence that is characteristic of spaces and fails in some □-sets

Def'n: A fibration between fibrant objects f is *fiberwise trivial* if its pullbacks along trivially fibrant objects are trivial:

$$\begin{array}{ccc} Y_x & \dashrightarrow & Y \\ {\scriptstyle \wr}\downarrow & \lrcorner & \downarrow f \\ K & \xrightarrow[x]{} & X \end{array} \qquad \text{for all } K \xrightarrow{\sim} 1,\ x\colon K \to X$$

Def'n: Say **FTFT** holds in a model category when all fiberwise trivial fibrations btw fibrant objects are trivial fibrations

Fiberwise triviality

Def'n: Say **FTFT** holds in a model category when all fiberwise trivial fibrations btw fibrant objects are trivial fibrations

Write $\mathbf{FTFT}_{-1}$ for property restricted to *propositional* fibrations ($f : Y \twoheadrightarrow X$ such that $\Delta_Y : Y \xrightarrow{\sim} Y \times_X Y$)

Th'm: These are invariant under Quillen equivalence

- ⊗ No surprise for experts; in $(\infty, 1)$-cat language they say: "if every pullback of (mono) f along $x : 1 \to X$ is iso, then f is iso"
- ⊗ In paper we also look at excluded middle; skipping today

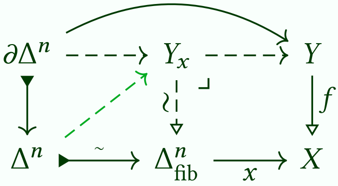
Fiberwise triviality

- In simplicial sets, let f be fiberwise trivial:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \Delta^n & \longrightarrow & X \end{array}$$

Fiberwise triviality

⊗ In simplicial sets, let f be fiberwise trivial:



⊗ So spaces have **FTFT**

⊗ Even holds constructively in constructive Kan–Quillen model structure of Henry '19, Gambino–Sattler–Szumiło '22

Fiberwise triviality

⊗ Intuition by looking at discrete model categories

$$\rightarrowtriangle \;=\; \rightarrowtail \;=\; \text{all maps} \qquad\qquad \xrightarrow{\sim} \;=\; \text{isomorphisms}$$

Th'm: The following are equivalent in a discrete model cat $\mathbb{C}$:

1. **FTFT**
2. **FTFT**$_{-1}$
3. $\mathbb{C}(1,-)$ is *conservative* ($\mathbb{C}(1, f)$ iso $\implies$ f iso)

⊗ A 1-topos where this holds and $0 \not\cong 1$ is called *well-pointed*

⊗ Any well-pointed Grothendieck topos is **Set**

Fiberwise triviality

- Example: n-truncated simplicial sets have **FTFT**

- Exotic example: *parameterized spectra* $\int_X \mathbf{Spectra}_X$ (which present a Grothendieck ∞-topos) has **FTFT**

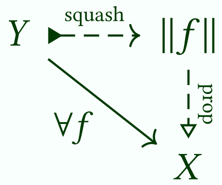
- Exotic example: $\mathbf{Set} \times \mathbf{Spectra}$ has $\mathbf{FTFT}_{-1}$ but not **FTFT** (don't know an ∞-topos example)

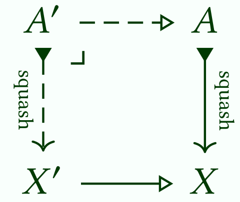
With propositional truncation

Def'n: Say a cofibration is *squash* if it left lifts against propositional fibrations

Def'n: Say a model category with pullback-stable cofibrations has a *stable propositional truncation* when

1. maps with fibrant codomain have (squash, prop) factorizations
2. squash maps with fibrant codomain preserved by pullback along fibrations



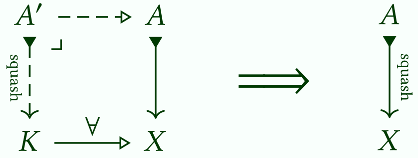


⊗ In discrete model categories: pullback-stable images

With propositional truncation

Th'm: The following are equivalent in a model cat with stable propositional truncations:

1. **FTFT**$_{-1}$
2. every *fiberwise squash cofibration* with fibrant codomain is squash:



where $K \xrightarrow{\sim} 1$

⊗ We'll use this characterization

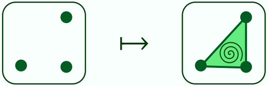
Table of contents

Outline

- For concreteness, look at cartesian cubes ($\otimes$ is $\times$)
- Candidate pathological object:

$$\mathbf{I}^2/\sigma = \mathrm{colim}\left\{ \mathbf{I}^2 \circlearrowleft \sigma \right\}$$

- We'll show that



$$1 + 1 + 1 \rightarrowtail \mathbf{I}^2/\sigma \overset{\sim}{\rightarrowtail} (\mathbf{I}^2/\sigma)_{\mathrm{fib}}$$

  is fiberwise squash but not squash

- Intuition: fiberwise squash maps only can't add points,
  but squash maps also can't add $\mathbf{I}^2/\sigma$'s

First half

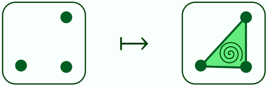
Lemma: Given

$$A \rightarrowtail B \overset{\sim}{\rightarrowtail} B_{\mathrm{fib}}$$

if $A \rightarrowtail B$ surjective on points then composite is fiberwise squash.

Proof: Depends on details—any point in B_{fib} connects to one in B

Instantiate with our candidate map:



$$1 + 1 + 1 \rightarrowtail \mathbf{I}^2/\sigma \overset{\sim}{\rightarrowtail} (\mathbf{I}^2/\sigma)_{\mathrm{fib}}$$

Second half

- Idea: squashing doesn't add new isolated points

  Lemma: If $A \rightarrowtail B \rightarrowtail B \sqcup C$ is squash, then C is empty.

- Want to see squashing *also* doesn't add new "isolated" $\mathbf{I}^2/\sigma$'s

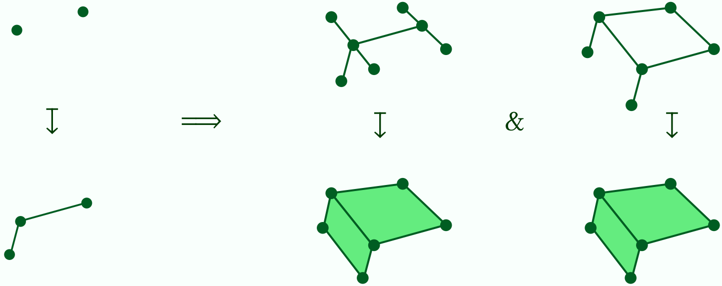
- **Strategy:** show $[\mathbf{I}^2/\sigma, -]$ preserves squash cofibrations

$$A \xrightarrow{\text{squash}} B \quad \Longrightarrow \quad [\mathbf{I}^2/\sigma, A] \xrightarrow{\text{squash}} [\mathbf{I}^2/\sigma, B]$$

points in here
are $\mathbf{I}^2/\sigma$'s in here

Square quotients in squash cofibrations

- ⊗ **To show:** $[\mathbf{I}^2/\sigma, -]$ preserves squash cofibrations
- ⊗ Use concrete description of squashing in □-sets: generated by
  - ⊘ open box inclusions $(A \times \mathbf{I}) \cup B \rightarrowtail B \times \mathbf{I}$
  - ⊘ boundary inclusions $(A \times \mathbf{I}) \cup (B \sqcup B) \rightarrowtail B \times \mathbf{I}$



Square quotients in squash cofibrations

- **To show:** $[\mathbf{I}^2/\sigma, -]$ preserves squash cofibrations
- Use concrete description of squashing in □-sets: generated by
  - open box inclusions $(A \times \mathbf{I}) \cup B \rightarrowtail B \times \mathbf{I}$
  - boundary inclusions $(A \times \mathbf{I}) \cup (B \sqcup B) \rightarrowtail B \times \mathbf{I}$
- Small object argument: every squash cofibration $\rightarrowtail$ is
  - a retract of...
  - a transfinite composite of...
  - pushouts of...
  - generating squash cofibrations.

⚠ **classical!**

"squash cofibration = composite of steps where we attach fillers"

Square quotients in squash cofibrations

⊗ Small object argument: every squash cofibration $\rightarrowtail$ is

- ⊘ a retract of... – preserved by $[\mathbf{I}^2/\sigma, -]$ (and any functor)
- ⊘ a transfinite composite of... – preserved by $[\mathbf{I}^2/\sigma, -]$

  $[\mathbf{I}^2, -]$ preserves all colimits (tiny object)

  compact objects (A s.t. $[A, -]$ preserves colims like these)
  closed under finite colimits

- ⊘ pushouts of... – preserved by $[\mathbf{I}^2/\sigma, -]$!

  A such that $[A, -]$ preserves pushouts along $\rightarrowtail$
  closed under finite monoid colimits

- ⊘ generating squash cofibrations.

  reduces to checking $[\mathbf{I}^2/\sigma, \mathbf{I}]$ contractible

Square quotients in squash cofibrations

- ⊘ generating squash cofibrations.

  reduces to checking $[\mathbf{I}^2/\sigma, \mathbf{I}]$ contractible

$[\mathbf{I}^2/\sigma, \mathbf{I}]$:

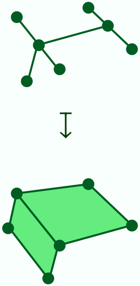
- $(i, j) \mapsto 0$
- $(i, j) \mapsto 1$
- ✖ $(i, j) \mapsto i$
- ✖ $(i, j) \mapsto j$

$(A \times \mathbf{I}) \cup B$

$\sim$

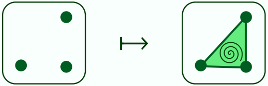
$\mathbf{I}^2/\sigma \longrightarrow B \times \mathbf{I}$

must be constant in this coordinate



Putting it all together

- We know that



$$1+1+1 \rightarrowtail \mathbf{I}^2/\sigma \overset{\sim}{\rightarrowtail} (\mathbf{I}^2/\sigma)_{\mathrm{fib}}$$

is fiberwise squash

- To show it's not squash, now suffices to show

$$[\mathbf{I}^2/\sigma, 1+1+1] \rightarrowtail [\mathbf{I}^2/\sigma, (\mathbf{I}^2/\sigma)_{\mathrm{fib}}]$$

is not squash

Putting it all together

- To show it's not squash, now suffices to show

$$[\mathbf{I}^2/\sigma, 1+1+1] \rightarrowtail [\mathbf{I}^2/\sigma, (\mathbf{I}^2/\sigma)_{\mathrm{fib}}] \sim [\mathbf{I}^2/\sigma, \mathbf{I}^2/\sigma]_{\mathrm{fib}}$$

  is not squash

- maps $\mathbf{I}^2/\sigma \to 1+1+1$ are constants

- maps $\mathbf{I}^2/\sigma \to \mathbf{I}^2/\sigma$ are

constants:	identity:	that's it:
$(i,j) \mapsto (0,0)$	$(i,j) \mapsto (i,j)$	✖ $(i,j) \mapsto (i,0)$

Putting it all together

- To show it's not squash, now suffices to show

$$[\mathbf{I}^2/\sigma, 1+1+1] \rightarrowtail [\mathbf{I}^2/\sigma, (\mathbf{I}^2/\sigma)_{\mathrm{fib}}] \sim [\mathbf{I}^2/\sigma, \mathbf{I}^2/\sigma]_{\mathrm{fib}}$$

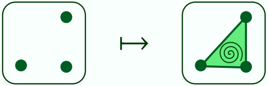
is not squash

- maps $\mathbf{I}^2/\sigma \to 1+1+1$ are constants
- maps $\mathbf{I}^n \times \mathbf{I}^2/\sigma \to \mathbf{I}^2/\sigma$ are

constants:	identity:	that's it.
$\vec{k}, (i, j) \mapsto f(\vec{k})$	$\vec{k}, (i, j) \mapsto (i, j)$	

$[\mathbf{I}^2/\sigma, \mathbf{I}^2/\sigma] \cong \mathbf{I}^2/\sigma + 1$ $\leftarrow$ point outside image of $1 + 1 + 1$!

Summarizing



$$1+1+1 \rightarrowtail \mathbf{I}^2/\sigma \overset{\sim}{\rightarrowtail} (\mathbf{I}^2/\sigma)_{\mathrm{fib}}$$

- $\otimes$ Doesn't add points $\implies$ fiberwise squash
- $\otimes$ Adds a new $\mathbf{I}^2/\sigma \implies$ not squash!

$\implies$ cartesian box-filling model structure fails **FTFT**

$\implies$ cartesian box-filling model structure is not spaces.

Other cube categories

- This version works in
  - cartesian cubical sets
  - affine cubical sets

$$\mathbf{I}^2/\sigma = \operatorname{colim}\left\{ \mathbf{I}^2 \circlearrowleft \sigma \right\}$$

- But not with connections!

$$\left.\begin{array}{c} \mathbf{I}^2/\sigma \times \mathbf{I} \to \mathbf{I}^2/\sigma \\ (i, j), t \mapsto (i \vee t, j \vee t) \end{array}\right\} \quad \begin{array}{c} \mathrm{id} \in [\mathbf{I}^2/\sigma, \mathbf{I}^2/\sigma] \text{ is not isolated} \\ \text{but contracts to a constant} \end{array}$$

- Can use a different quotient in
  - De Morgan cubical sets
  - boolean cubical sets

$$\mathbf{I}/\neg = \operatorname{colim}\left\{ \mathbf{I} \circlearrowleft \neg \right\}$$

Other cube categories

- For box-filling model structures:

Affine (BCH)	$\delta, \varepsilon, \sigma$	✗
Cartesian (AFH+ABCFHL)	$\delta, \varepsilon, \Delta, \sigma$	✗
Semilattice (CS)	$\delta, \varepsilon, \Delta, \sigma, \vee,$	✓
Dedekind	$\delta, \varepsilon, \Delta, \sigma, \vee, \wedge$	?
De Morgan (CCHM)	$\delta, \varepsilon, \Delta, \sigma, \vee, \wedge, \neg$	✗

- Equivariant model structure fixes "cartesian" with more complicated open boxes—make $\mathbf{I}^n/G$ contractible

Closing remarks

- Christian recently found an explicit construction of a non-trivial fiberwise trivial fibration in these cases

  Wait for the paper ☺

- Hints towards characterizations of the model structures?

  At least for cartesian cubes, think so (WIP!)

- Do the equivariant and one-connection model structures validate **FTFT** *constructively*?

  We are doubtful...

Thank you!