Cubical Indexed Inductive Types

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jww Robert Harper

Higher inductive types

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- roll quotients and inductive types into one

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$$\begin{aligned}
&\mathbf{data}\ \mathrm{int}\ \mathbf{where}\\
&\mid \mathrm{neg}(n : \mathrm{nat}) : \mathrm{int}\\
&\mid \mathrm{pos}(n : \mathrm{nat}) : \mathrm{int}\\
&\mid \mathrm{seg} : \mathrm{Id}_{\mathrm{int}}(\mathrm{neg}(0), \mathrm{pos}(0))
\end{aligned}$$

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- Useful generality beyond ordinary quotients

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- Useful generality beyond ordinary quotients
  - → In ordinary TT: Cauchy reals, QIITs, . . .

Higher inductive types

- roll quotients and inductive types into one

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- Useful generality beyond ordinary quotients
  - → In ordinary TT: Cauchy reals, QIITs, . . .
  - → In higher-d TT: truncations, . . .

Higher inductive types: what are they?

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- HoTT: idea, many examples
  - → circles, pushouts, . . . ("ordinary" quotients)
  - → truncations, localizations (higher inductive types)
  - → Cauchy reals (higher inductive-inductive types)

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- **no precise syntactic schema**
  - → what forms can constructors take?
  - → deriving eliminators

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- HoTT: idea, many examples
  - → circles, pushouts, . . . ("ordinary" quotients)
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  - → Cauchy reals (higher inductive-inductive types)
- **no precise syntactic schema**
  - → what forms can constructors take?
  - → deriving eliminators
- **missing semantics for many**
  - → how do programs with HITs compute?
  - → which HITs exist in denotational models? (pushouts?)

Higher inductive types: what are they?

- advances in syntax
  - → Sojakova 2014: W-suspensions
  - → Basold, Geuvers, & van der Weide 2017: 1-d HITs
  - → Dybjer & Moeneclaey 2017: 2-d HITs
  - → Kaposi & Kovács 2018: n-d HIITs

Higher inductive types: what are they?

- advances in semantics
  - → Lumsdaine & Shulman 2017: simplicial model cats
    - → syntax? HITs with parameters?
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- the "edge of understanding": syntax + semantics
  - ⓐ DM, AKK (truncated)
  - ⓑ **cubical type theories**

Cubical type theories

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- Motive: higher type theories w/ computational meaning

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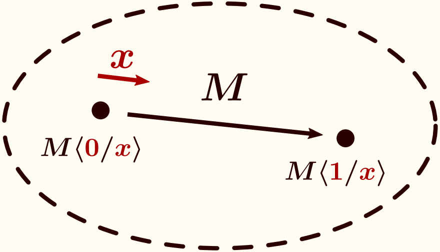
- Several varieties:
  - → De Morgan cubes
    Cohen, Coquand, Huber, & Mörtberg 2015
  - → Cartesian cubes
    Angiuli, Favonia, & Harper 2018
    Angiuli, Brunerie, Coquand, Favonia, Licata, & Harper 2019
  - → Substructural cubes
    Bezem, Coquand, & Huber 2013&2017

Cubical type theories

- Motive: higher type theories w/ computational meaning

$$M \in A\ [\Psi, x]$$



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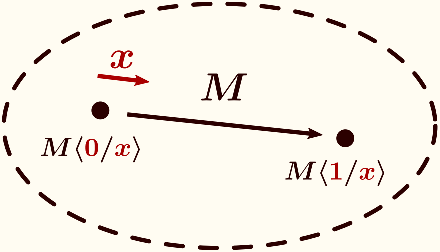

$$\lambda^{\mathbb{I}} x.M \in \mathrm{Path}_{x.A}(M\langle 0/x \rangle, M\langle 1/x \rangle) \; [\Psi]$$

Cubical type theories

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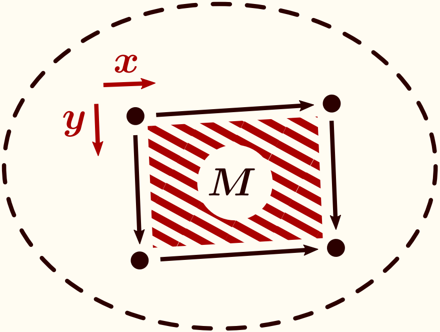
$$M \in A\ [\Psi, x, y]$$



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- **Useful for practical formalization**
  - → early motivation: Licata & Brunerie 2015

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- Developed to give computational meaning to HoTT
  - → also: constructive denotational models
  - → also: *any* models for many HITs
- Useful for practical formalization
  - → early motivation: Licata & Brunerie 2015
- For HoTT purists: laboratory of higher ideas

Cubical inductive types: syntax

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- Higher constructors via dimension arguments

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- Path constructors live in the inductive type
- Scales well to >1-d (spheres, torus, . . .)

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$$\mathbf{elim}\ Z$$
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- reduces when applied to a constructor

$$\mathbf{elim}\ (\mathrm{seg}(y))\ [\cdots] \ = \ M_0\langle y/x\rangle$$

Cubical inductive types: syntax

- Dealing with higher dimensions

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- Dealing with higher dimensions
  - → HITs with >1-d constructors

$$\textbf{data}\ \text{sphere}\ \textbf{where}$$
$$|\ \text{base} : \text{sphere}$$
$$|\ \text{surf}(x : \mathbb{I}, y : \mathbb{I}) : \text{sphere}$$
$$[x = 0 \mid x = 1 \mid y = 0 \mid y = 1 \hookrightarrow \text{base}]$$

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  - → pattern-matching on two HITs at once

$$\textbf{elim}\ Z_1, Z_2$$
$$|\ \text{seg}(x), \text{seg}(y) \to M_{00}$$
$$\vdots$$

Cubical inductive types: syntax

- C & Harper: schema for indexed cubical HITs

$$
\begin{aligned}
&\textbf{data } \mathrm{X} \textbf{ where}\\
&\mid \mathrm{constr}_1\\
&\quad (a_1 : A_1) \cdots (a_k : A_k)\\
&\quad (b_1 : \mathrm{B}_1) \cdots (b_k : \mathrm{B}_k)\\
&\quad (x_1, \ldots, x_\ell : \mathrm{dim})\\
&\quad [r_1 = r_1' \hookrightarrow \mathrm{M}_1 \mid \cdots \mid r_j = r_j' \hookrightarrow \mathrm{M}_j]\\
&\mid \mathrm{constr}_n \cdots
\end{aligned}
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$$\mathrm{B} ::= \mathbf{X} \mid (a : A) \to \mathrm{B}$$

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\end{array}$$

$$\mathrm{B} ::= \mathbf{X} \mid (a : A) \to \mathrm{B}$$

$$\mathrm{M} ::= b \mid \mathrm{constr}_i(\vec{M}, \vec{\mathrm{M}}, \vec{r}) \mid \mathrm{hcom}(\cdots) \mid \lambda a.\mathrm{M} \mid \mathrm{M} M$$

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- Ingredients of a cubical type

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  - → Values

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$$\langle M, N \rangle \in A \times B$$
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  - → Coercion and composition operations

Interlude: semantics of cubical type theory

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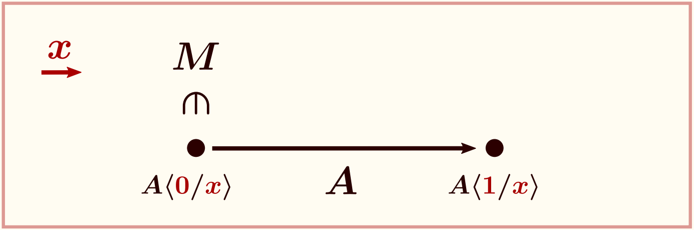

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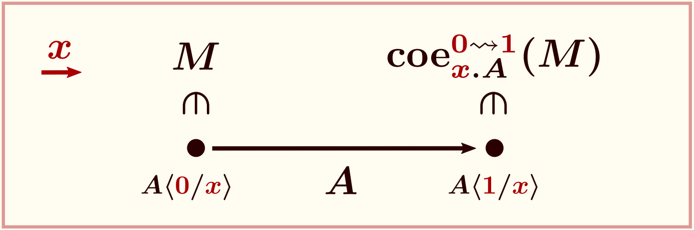

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- ⥁ Coercion: all constructions respect paths
  - → Evaluate by cases on the type line

Interlude: semantics of cubical type theory

- ⟳ Coercion: all constructions respect paths

  → Evaluate by cases on the type line

$$\mathbf{coe}^{r \rightsquigarrow s}_{x.A \times B}(\langle M, N \rangle)$$

$$\longmapsto$$

$$\langle \mathbf{coe}^{r \rightsquigarrow s}_{x.A}(M), \mathbf{coe}^{r \rightsquigarrow s}_{x.B}(N) \rangle$$

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- ↻ Coercion: all constructions respect paths
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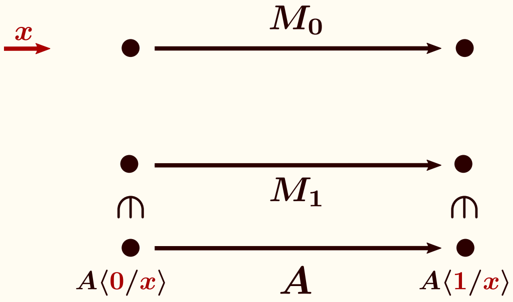

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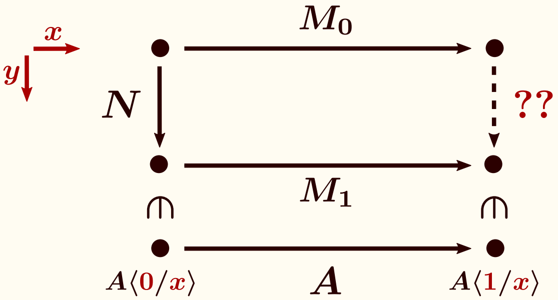

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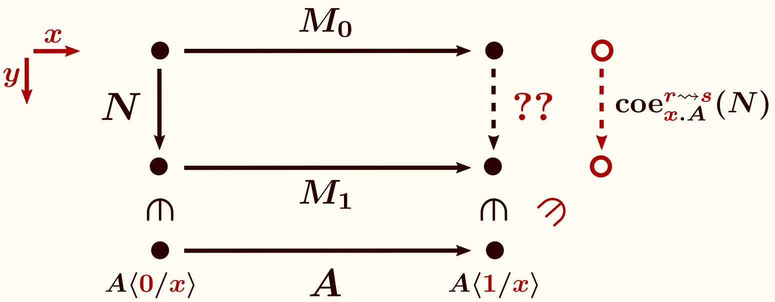

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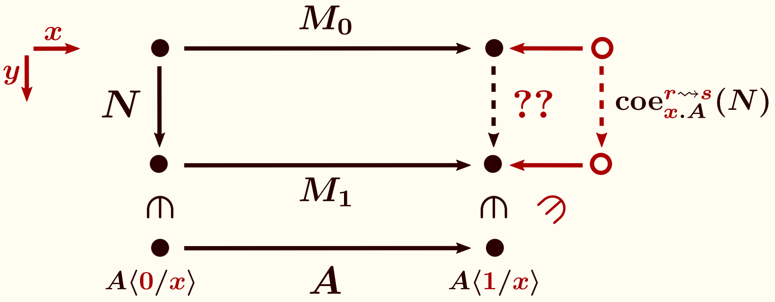

Interlude: semantics of cubical type theory

- **Composition:** strengthening the induction hypothesis

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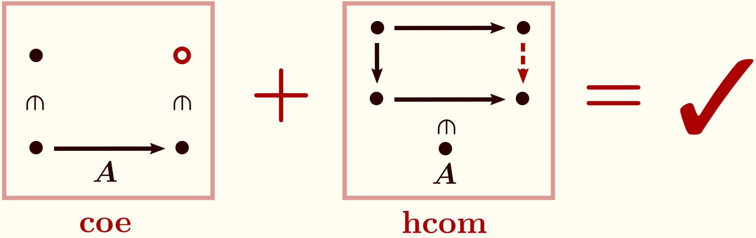

Interlude: semantics of cubical type theory

↻ Composition: strengthening the induction hypothesis

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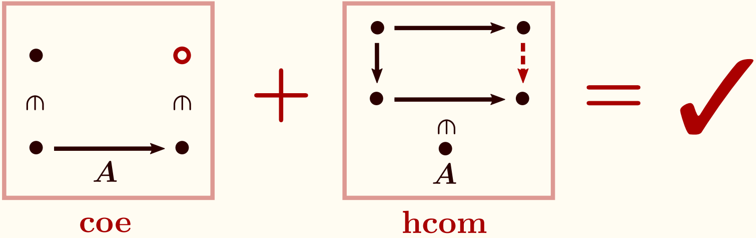

(ⓑ generalize **coe** to heterogeneous composition)

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- What are the values of a higher inductive type?

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$$\mid \text{rel}(a : A, b : A, u : R\langle a, b\rangle, x : \mathbb{I}) : \text{quo}$$
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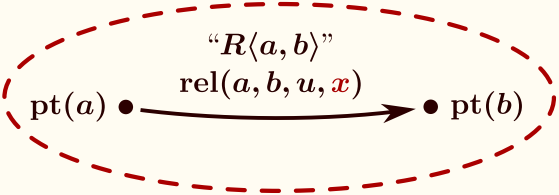

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$$\text{pt}(M)\ \textbf{val} \qquad \text{rel}(M, N, P, x)\ \textbf{val}$$
$$\left(\begin{array}{l} \text{rel}(M, N, P, 0) \longmapsto \text{pt}(M) \\ \text{rel}(M, N, P, 1) \longmapsto \text{pt}(N) \end{array}\right)$$

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$$\mathrm{pt}(M)\ \textbf{val} \qquad \frac{\mathrm{rel}(M, N, P, x)\ \textbf{val}}{\begin{pmatrix} \mathrm{rel}(M, N, P, 0) \longmapsto \mathrm{pt}(M) \\ \mathrm{rel}(M, N, P, 1) \longmapsto \mathrm{pt}(N) \end{pmatrix}}$$

- Can we implement coercion and composition?

Cubical inductive types: semantics

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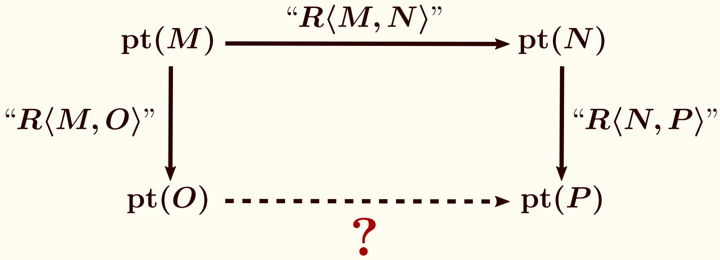

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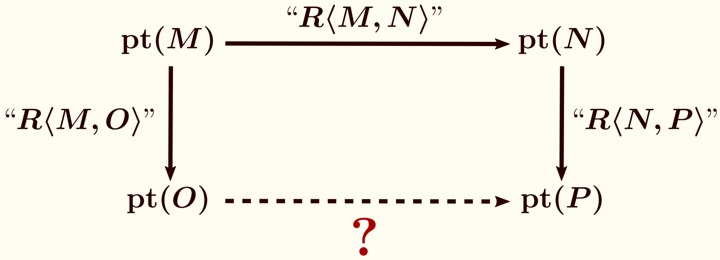

- Unless R is symmetric and transitive, ? may not exist

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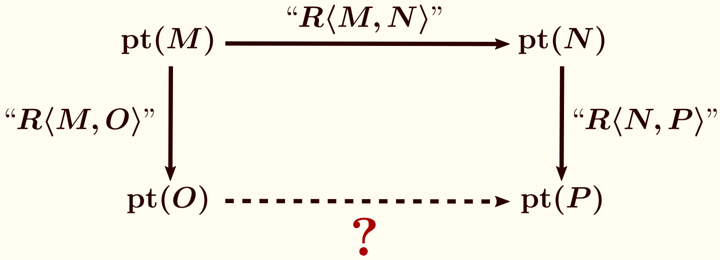

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  ⇒ Must revise our choice of values

Cubical inductive types: semantics

- Idea: freely add homogeneous composition values

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$$\begin{pmatrix} \mathrm{rel}(M, N, P, 0) \longmapsto \mathrm{pt}(M) \\ \mathrm{rel}(M, N, P, 1) \longmapsto \mathrm{pt}(N) \end{pmatrix}$$

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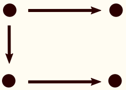
hcom () **val**

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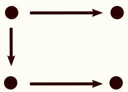
hcom () **val**

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- Eliminator maps $\mathbf{hcom}$ values to $\mathbf{hcom}$s in the target

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$$\frac{A \in \mathcal{U}\ [\Psi, x] \qquad R \in A \times A \to \mathcal{U}\ [\Psi, x] \qquad M \in \mathrm{quo}(A,R)\langle r/x\rangle\ [\Psi]}{\mathrm{coe}^{r\rightsquigarrow s}_{x.\mathrm{quo}(A,R)}(M) \in \mathrm{quo}(A,R)\langle s/x\rangle\ [\Psi]}$$

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→ cf. Lumsdaine & Shulman

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Theorem (Canonicity).
Any term in a HIT evaluates to a value.

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→ any zero-d $\mathbf{int}$ evaluates to $\mathbf{pos}$ or $\mathbf{neg}$ ✓

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  - → Size depends on size of indices, but not parameters

Cubical type theory and the future of HITs

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- Going further
  - → inductive-inductive types (Hugunin 2019)
  - → implementation (`redtt`, Cubical Agda)
  - → more flexible schemata