Higher Inductive Types in Computational Cubical Type Theory

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cubical type theory

dependent type theory with a
univalent, proof-relevant internal equality

cubical type theory

dependent type theory with a
univalent, proof-relevant internal equality

$\Uparrow$

indexed higher inductive types

- **quotient types for this equality**
- **indexed inductive types that respect it**

higher type theory:

[Awodey & Warren; Voevodsky]

dependent type theory with a
univalent, proof-relevant internal equality

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dependent type theory with a

univalent, proof-relevant internal equality

$$\frac{M_0 \in A \qquad M_1 \in A}{\texttt{Path}_A(M_0, M_1)\ \texttt{type}}$$

higher type theory:

[Awodey & Warren; Voevodsky]

dependent type theory with a

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"$\texttt{Path}_A(M_0, M_1) \to B M_0 \to B M_1$"

higher type theory:

[Awodey & Warren; Voevodsky]

dependent type theory with a

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isomorphism ⇒ equal types

"$\texttt{Path}_A(M_0, M_1) \to B M_0 \to B M_1$"

higher type theory:

[Awodey & Warren; Voevodsky]

dependent type theory with a

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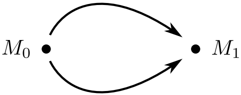

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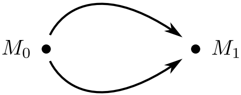

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isomorphism ⇒ equal types

"$\texttt{Path}_A(M_0, M_1) \to BM_0 \to BM_1$"

(axiomatized by homotopy type theory)

cubical type theory:

[Cohen, Coquand, Huber & Mörtberg; Angiuli, Favonia & Harper]

computational higher type theory via dimension variables

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computational higher type theory via dimension variables

$A\ \texttt{type}\ [x_1, \ldots, x_n] \qquad M \in A\ [x_1, \ldots, x_n]$

cubical type theory:

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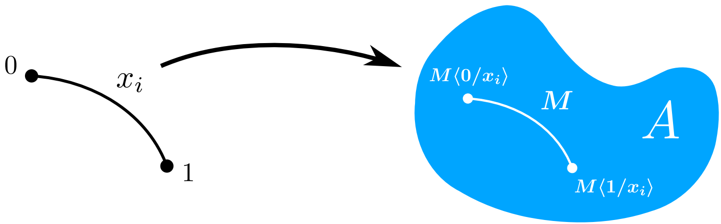

cubical type theory:

path types

$$\frac{M \in A\ [\Psi, x]}{\lambda^{\mathbb{I}} x.M \in \mathtt{Path}_A(M\langle 0/x\rangle, M\langle 1/x\rangle)\ [\Psi]}$$

$$\frac{P \in \mathtt{Path}_A(M_0, M_1)\ [\Psi] \qquad r \in \Psi \cup \{0, 1\}}{P@r \in A\ [\Psi]}$$

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$$\lambda^{\mathbb{I}} x.M\ \mathtt{val} \qquad\qquad (\lambda^{\mathbb{I}} x.M)@r \longmapsto M\langle r/x\rangle$$

cubical type theory:

coercion

$$\frac{C \ \texttt{type} \ [\Psi, x] \qquad N \in C\langle 0/x \rangle \ [\Psi]}{\texttt{coe}_{x.C}^{0 \rightsquigarrow 1}(N) \in C\langle 1/x \rangle \ [\Psi]}$$

cubical type theory:

coercion

$$\frac{C\ \texttt{type}\ [\Psi, x] \qquad N \in C\langle r/x\rangle\ [\Psi] \qquad r, s \in \Psi \cup \{0,1\}}{\texttt{coe}_{x.C}^{r \rightsquigarrow s}(N) \in C\langle s/x\rangle\ [\Psi]}$$

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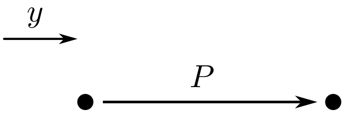
$$\texttt{coe}^{r\rightsquigarrow s}_{x.A\times B}(N) \longmapsto \langle \texttt{coe}^{r\rightsquigarrow s}_{x.A}(\texttt{fst}(N)), \texttt{coe}^{r\rightsquigarrow s}_{x.B}(\texttt{snd}(N))\rangle$$

$$\texttt{coe}^{r\rightsquigarrow s}_{x.A\to B}(N) \longmapsto \lambda a.\texttt{coe}^{r\rightsquigarrow s}_{x.B}(N(\texttt{coe}^{s\rightsquigarrow r}_{x.A}(a)))$$

$$\texttt{coe}^{r\rightsquigarrow s}_{x.\texttt{Path}_A(M_0,M_1)}(N) \longmapsto ?$$

cubical type theory:

composition



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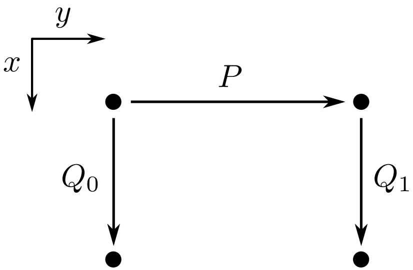

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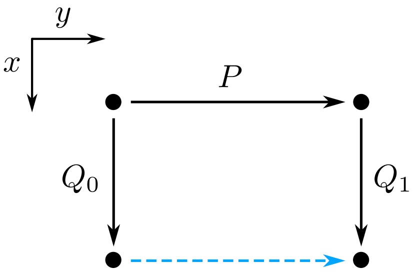

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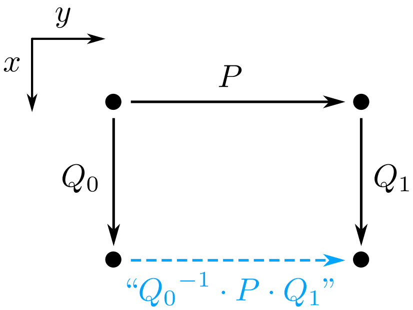

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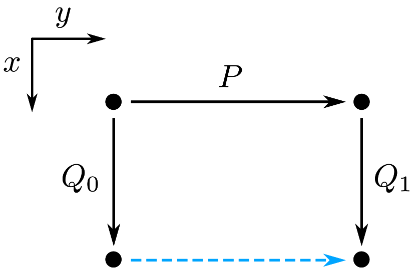
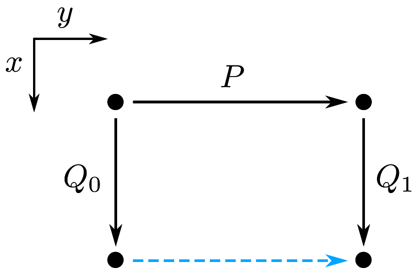


$\mathtt{hcom}_A^{0 \rightsquigarrow 1}(P; y = 0 \hookrightarrow x.Q_0, y = 1 \hookrightarrow x.Q_1)$

cubical type theory:

composition



$\mathtt{hcom}_A^{0 \rightsquigarrow 1}(P; y = 0 \hookrightarrow x.Q_0, y = 1 \hookrightarrow x.Q_1)$

general case: $\mathtt{hcom}_A^{r \rightsquigarrow s}(M; \overrightarrow{r_i = r_i' \hookrightarrow x.N_i})$

cubical type theory:

+ univalence

higher inductive types

quotients for proof-relevant equality

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```
data nat where
| zero
| suc (n : nat)
```

higher inductive types

quotients for proof-relevant equality

```
data nat where
| zero
| suc (n : nat)
```

```
data int where
| pos (n : nat)
| neg (n : nat)
| "neg(zero) = pos(zero)"
```

higher inductive types

quotients for proof-relevant equality

$\texttt{data nat where}$
$\mid \texttt{zero}$
$\mid \texttt{suc}\ (n : \texttt{nat})$

$\texttt{data int where}$
$\mid \texttt{pos}\ (n : \texttt{nat})$
$\mid \texttt{neg}\ (n : \texttt{nat})$
$\mid \texttt{seg}\ (x : \mathbb{I})\ [x = 0 \hookrightarrow \texttt{neg}(\texttt{zero}), x = 1 \hookrightarrow \texttt{pos}(\texttt{zero})]$

higher inductive types

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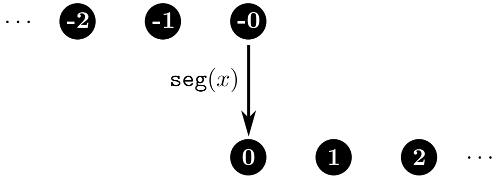
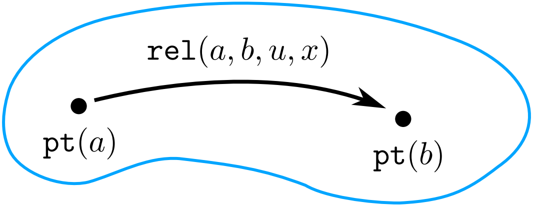


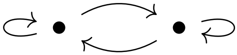
higher inductive types



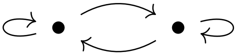
$$
\begin{aligned}
&A : \texttt{type}, R : A \times A \to \texttt{type} \vdash \texttt{data quo where}\\
&\mid \texttt{pt}\ (a : A)\\
&\mid \texttt{rel}\ (a, b : A)(u : R\langle a, b\rangle)(x : \mathbb{I})\\
&\quad [x = 0 \hookrightarrow \texttt{pt}(a), x = 1 \hookrightarrow \texttt{pt}(b)]
\end{aligned}
$$

higher inductive types

$\mathtt{quo}(\mathtt{bool}, \top) \quad \Rightarrow$

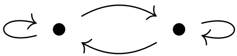


higher inductive types

$\texttt{quo}(\texttt{bool}, \top) \quad \Rightarrow$ 

$A : \texttt{type} \vdash \texttt{data trunc where}$
$\mid \texttt{pt}\ (a : A)$
$\mid \texttt{squash}\ (t_0\ t_1 : \texttt{trunc})(x : \mathbb{I})\ [x = 0 \hookrightarrow t_0, x = 1 \hookrightarrow t_1]$

higher inductive types

$\texttt{quo}(\texttt{bool}, \top) \quad \Rightarrow$ 

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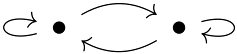
higher inductive types

$\mathtt{quo}(\mathtt{bool}, \top) \quad \Rightarrow$ 

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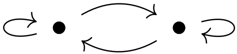
$\mathtt{trunc}(\mathtt{bool}) \quad \Rightarrow$ 

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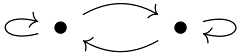
$\texttt{quo}(\texttt{bool}, \top) \quad \Rightarrow$ 

$A : \texttt{type} \vdash$ **data** $\texttt{trunc}$ **where**
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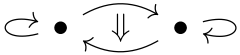
$\texttt{trunc}(\texttt{bool}) \quad \Rightarrow$ 

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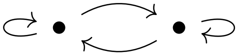
$\texttt{quo}(\texttt{bool}, \top) \quad \Rightarrow$ 

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$\texttt{trunc}(\texttt{bool}) \quad \Rightarrow$ **etc.**

higher inductive types

in generality

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- **axiomatic type theory:**
  - Sojakova: W-quotients
  - Basold, Geuvers, & van der Weide; Dybjer & Moeneclaey; Kaposi & Kovács
- **semantics:**
  - Dybjer & Moeneclaey
  - Lumsdaine & Shulman: cell monads
- **cubical type theory:**
  - Coquand, Huber, & Mörtberg: examples, schema sketch

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our contribution:

cubical schema with computational semantics

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our contribution:

cubical schema with computational semantics

(including indexed inductive types)

1. schema

$$
\begin{array}{l}
\Gamma \vdash \texttt{data}\ \texttt{X}\ \texttt{where} \\
\mid \texttt{intro}_1\ (a_1 : A_1) \cdots (a_m : A_m) \\
\quad (b_1 : \mathrm{B}_1) \cdots (b_k : \mathrm{B}_k) \\
\quad (x_1, \ldots, x_\ell : \mathbb{I}) \\
\quad [r_1 = r_1' \hookrightarrow \mathrm{M}_1, \ldots, r_j = r_j' \hookrightarrow \mathrm{M}_j] \\
\vdots \\
\mid \texttt{intro}_n\ \cdots
\end{array}
$$

$$\mathrm{B} ::= \texttt{X} \mid (a{:}A) \to \mathrm{B}$$

$$\mathrm{M} ::= b \mid \texttt{intro}_i(\vec{M}, \vec{\mathrm{M}}, \vec{r}) \mid \texttt{hcom}(\cdots) \mid \lambda a.\mathrm{M} \mid \mathrm{M}M$$

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$$

elimination principle ⇝

$$\mathrm{B} ::= \texttt{X} \mid (a{:}A) \to \mathrm{B}$$

$$\mathrm{M} ::= b \mid \texttt{intro}_i(\vec{M}, \vec{\mathrm{M}}, \vec{r}) \mid \texttt{hcom}(\cdots) \mid \lambda a.\mathrm{M} \mid \mathrm{M}M$$

2. semantics

what are the values of an inductive type?

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\end{array}
\rightsquigarrow
\begin{array}{l}
\texttt{pt}(M)\ \texttt{val} \\
\texttt{rel}(M, N, T, x)\ \texttt{val} \\
\texttt{rel}(M, N, T, 0) \longmapsto \texttt{pt}(M) \\
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can we implement coercion and composition?

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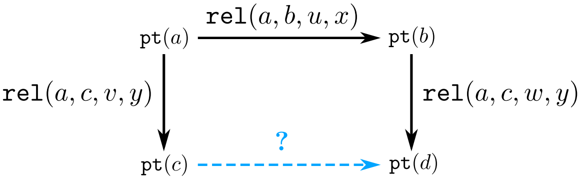

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$\rightsquigarrow$

$\texttt{pt}(M)$ $\texttt{val}$

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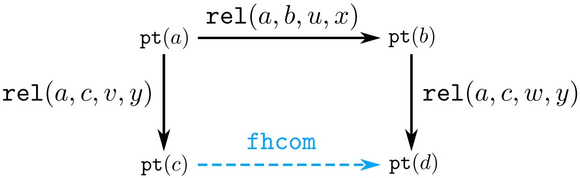

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can we implement coercion and composition?

$$\texttt{fhcom}^{r \rightsquigarrow s}(M; \overrightarrow{r_i = r'_i \hookrightarrow x.N_i})\ \texttt{val} \qquad \text{if } r \neq s,\ \forall i. r_i \neq r'_i$$
$$\texttt{fhcom}^{r \rightsquigarrow s}(M; \overrightarrow{r_i = r'_i \hookrightarrow x.N_i}) \longmapsto N_i\langle s/x\rangle \qquad \text{if } r_i = r'_i,\ \forall j < i. r_j \neq r'_j$$
$$\texttt{fhcom}^{r \rightsquigarrow s}(M; \overrightarrow{r_i = r'_i \hookrightarrow x.N_i}) \longmapsto M \qquad \text{if } r = s,\ i. r_i \neq r'_i$$

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$$\cdots \texttt{fhcom} \cdots$$

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$$\texttt{rel}(M, N, T, x)\ \texttt{val}$$
$$\texttt{rel}(M, N, T, 0) \longmapsto \texttt{pt}(M)$$
$$\texttt{rel}(M, N, T, 1) \longmapsto \texttt{pt}(N)$$
$$\cdots \texttt{fhcom} \cdots$$

can we implement coercion and composition? and elimination?

2. semantics

what are the values of an inductive type?

$$A, R \vdash \texttt{data}\ \texttt{quo}\ \texttt{where}$$
$$\mid \texttt{pt}\ (a : A)$$
$$\mid \texttt{rel}\ (a, b : A)(u : R\langle a, b\rangle)(x : \mathbb{I})$$
$$[x = 0 \hookrightarrow \texttt{pt}(a), x = 1 \hookrightarrow \texttt{pt}(b)]$$

$\rightsquigarrow$

$$\texttt{pt}(M)\ \texttt{val}$$
$$\texttt{rel}(M, N, T, x)\ \texttt{val}$$
$$\texttt{rel}(M, N, T, 0) \longmapsto \texttt{pt}(M)$$
$$\texttt{rel}(M, N, T, 1) \longmapsto \texttt{pt}(N)$$
$$\cdots \texttt{fhcom} \cdots$$

can we implement coercion and composition? and elimination?

$$\texttt{rec}_C(\texttt{pt}(M); a.P, a.b.u.x.Q) \longmapsto P[M/a]$$
$$\texttt{rec}_C(\texttt{rel}(M, N, T, y); a.P, a.b.u.x.Q) \longmapsto Q[M, N, T/a, b, u]\langle y/x\rangle$$

2. semantics

what are the values of an inductive type?

$$A, R \vdash \texttt{data quo where} \\ \mid \texttt{pt}\ (a : A) \\ \mid \texttt{rel}\ (a, b : A)(u : R\langle a, b\rangle)(x : \mathbb{I}) \\ \quad [x = 0 \hookrightarrow \texttt{pt}(a), x = 1 \hookrightarrow \texttt{pt}(b)]$$

$\rightsquigarrow$

$$\texttt{pt}(M)\ \texttt{val} \\ \texttt{rel}(M, N, T, x)\ \texttt{val} \\ \texttt{rel}(M, N, T, 0) \longmapsto \texttt{pt}(M) \\ \texttt{rel}(M, N, T, 1) \longmapsto \texttt{pt}(N) \\ \cdots \texttt{fhcom} \cdots$$

can we implement coercion and composition? and elimination?

$$\texttt{rec}_C(\texttt{pt}(M); a.P, a.b.u.x.Q) \longmapsto P[M/a]$$

$$\texttt{rec}_C(\texttt{rel}(M, N, T, y); a.P, a.b.u.x.Q) \longmapsto Q[M, N, T/a, b, u]\langle y/x\rangle$$

$$\texttt{rec}_C(\texttt{fhcom}^{r \rightsquigarrow s}(M; \overrightarrow{r_i = r_i' \hookrightarrow y.N_i}); \ldots) \longmapsto$$

$$\texttt{hcom}_C^{r \rightsquigarrow s}(\texttt{rec}_C(M; \ldots); \overrightarrow{r_i = r_i' \hookrightarrow y.\texttt{rec}_C(N_i; \ldots)})$$

2. semantics

what are the values of an inductive type?

$$\begin{array}{l} A, R \vdash \texttt{data quo where} \\ \mid \texttt{pt}\ (a : A) \\ \mid \texttt{rel}\ (a, b : A)(u : R\langle a, b\rangle)(x : \mathbb{I}) \\ \quad [x = 0 \hookrightarrow \texttt{pt}(a), x = 1 \hookrightarrow \texttt{pt}(b)] \end{array} \quad \rightsquigarrow \quad \begin{array}{l} \texttt{pt}(M)\ \texttt{val} \\ \texttt{rel}(M, N, T, x)\ \texttt{val} \\ \texttt{rel}(M, N, T, 0) \longmapsto \texttt{pt}(M) \\ \texttt{rel}(M, N, T, 1) \longmapsto \texttt{pt}(N) \\ \cdots \texttt{fhcom} \cdots \end{array}$$

can we implement coercion and composition? and elimination?

$$\texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(\texttt{pt}(M)) \quad \longmapsto \quad \texttt{pt}(\texttt{coe}^{r \rightsquigarrow s}_{x.A}(M))$$

$$\texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(\texttt{rel}(M, N, T, y)) \longmapsto$$

$$\texttt{rel}(\texttt{coe}^{r \rightsquigarrow s}_{x.A}(M), \texttt{coe}^{r \rightsquigarrow s}_{x.B}(N), \texttt{coe}^{r \rightsquigarrow s}_{x.\text{“}R\text{”}}(T), y)$$

2. semantics

what are the values of an inductive type?

$$A, R \vdash \texttt{data quo where} \\ \mid \texttt{pt}\ (a : A) \\ \mid \texttt{rel}\ (a, b : A)(u : R\langle a, b\rangle)(x : \mathbb{I}) \\ \quad [x = 0 \hookrightarrow \texttt{pt}(a), x = 1 \hookrightarrow \texttt{pt}(b)]$$

$\rightsquigarrow$

$$\texttt{pt}(M)\ \texttt{val} \\ \texttt{rel}(M, N, T, x)\ \texttt{val} \\ \texttt{rel}(M, N, T, 0) \longmapsto \texttt{pt}(M) \\ \texttt{rel}(M, N, T, 1) \longmapsto \texttt{pt}(N) \\ \cdots \texttt{fhcom} \cdots$$

can we implement coercion and composition? and elimination?

$$\texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(\texttt{pt}(M)) \quad \longmapsto \quad \texttt{pt}(\texttt{coe}^{r \rightsquigarrow s}_{x.A}(M))$$

$$\texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(\texttt{rel}(M, N, T, y)) \longmapsto * \\ \texttt{rel}(\texttt{coe}^{r \rightsquigarrow s}_{x.A}(M), \texttt{coe}^{r \rightsquigarrow s}_{x.B}(N), \texttt{coe}^{r \rightsquigarrow s}_{x.\text{“}R\text{”}}(T), y)$$

* **more complicated in general case**

2. semantics

what are the values of an inductive type?

$$\begin{array}{l} A, R \vdash \texttt{data}\ \texttt{quo}\ \texttt{where} \\ \mid \texttt{pt}\ (a : A) \\ \mid \texttt{rel}\ (a, b : A)(u : R\langle a, b\rangle)(x : \mathbb{I}) \\ \quad [x = 0 \hookrightarrow \texttt{pt}(a), x = 1 \hookrightarrow \texttt{pt}(b)] \end{array} \quad \rightsquigarrow \quad \begin{array}{l} \texttt{pt}(M)\ \texttt{val} \\ \texttt{rel}(M, N, T, x)\ \texttt{val} \\ \texttt{rel}(M, N, T, 0) \longmapsto \texttt{pt}(M) \\ \texttt{rel}(M, N, T, 1) \longmapsto \texttt{pt}(N) \\ \cdots \texttt{fhcom} \cdots \end{array}$$

can we implement coercion and composition? and elimination?

$$\texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(\texttt{fhcom}^{r \rightsquigarrow s}(M; \overrightarrow{r_i = r'_i \hookrightarrow N_i})) \longmapsto$$

$$\texttt{hcom}^{r \rightsquigarrow s}_{\texttt{quo}(A,R)\langle s/x\rangle}(\texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(M); \overrightarrow{r_i = r'_i \hookrightarrow \texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(N_i)})$$

2. semantics

what are the values of an inductive type?

$$A, R \vdash \texttt{data}\ \texttt{quo}\ \texttt{where}$$
$$\mid \texttt{pt}\ (a : A)$$
$$\mid \texttt{rel}\ (a, b : A)(u : R\langle a, b\rangle)(x : \mathbb{I})$$
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$\rightsquigarrow$

$\texttt{pt}(M)\ \texttt{val}$

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$\texttt{rel}(M, N, T, 0) \longmapsto \texttt{pt}(M)$

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$\cdots \texttt{fhcom} \cdots$

can we implement coercion and composition? and elimination?

$$\texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(\texttt{fhcom}^{r \rightsquigarrow s}(M; \overrightarrow{r_i = r_i' \hookrightarrow N_i})) \longmapsto$$
$$\texttt{fhcom}^{r \rightsquigarrow s}(\texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(M); \overrightarrow{r_i = r_i' \hookrightarrow \texttt{coe}^{r \rightsquigarrow s}_{x.\texttt{quo}(A,R)}(N_i)})$$

indexed inductive types

indexed inductive types

identity type (subject of HoTT axioms)

$$A : \texttt{type} \vdash \texttt{data}\ \texttt{Id}(a\ b : A)\ \texttt{where}$$
$$\mid \texttt{refl}\ (a : A) : \texttt{Id}(a, a)$$

indexed inductive types

identity type (subject of HoTT axioms)

$$A : \mathtt{type} \vdash \mathtt{data}\ \mathtt{Id}(a\ b : A)\ \mathtt{where}$$
$$\mid \mathtt{refl}\ (a : A) : \mathtt{Id}(a, a)$$

$$P \in \mathtt{Path}_A(M_0, M_1)$$

$$\Downarrow$$

$$\mathtt{coe}^{0 \rightsquigarrow 1}_{x.\mathtt{Id}(A)(M_0, P@x)}(\mathtt{refl}(M_0)) \longmapsto\ ?\ \in \mathtt{Id}(A)(M_0, M_1)$$

indexed inductive types

identity type (subject of HoTT axioms)

$$A : \texttt{type} \vdash \texttt{data}\ \texttt{Id}(a\ b : A)\ \texttt{where}$$
$$\mid \texttt{refl}\ (a : A) : \texttt{Id}(a, a)$$

$$P \in \texttt{Path}_A(M_0, M_1)$$
$$\Downarrow$$
$$\texttt{coe}^{0 \rightsquigarrow 1}_{x.\texttt{Id}(A)(M_0, P@x)}(\texttt{refl}(M_0)) \longmapsto \text{“}\texttt{fcoe}^{0 \rightsquigarrow 1}_{x.(M_0, P@x)}(\texttt{refl}(M_0))\text{”}$$

indexed inductive types

identity type (subject of HoTT axioms)

$$A : \mathtt{type} \vdash \mathtt{data}\ \mathtt{Id}(a\ b : A)\ \mathtt{where}$$
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$$\mathtt{fcoe}^{r\rightsquigarrow s}_{x.(-,-)}(-) \quad + \quad \mathtt{coe}^{r\rightsquigarrow s}_{x.A}(-) \quad \Rightarrow \quad \mathtt{coe}^{r\rightsquigarrow s}_{x.\mathtt{Id}_A(M_0, M_1)}(-)$$

indexed inductive types

identity type (subject of HoTT axioms) ✓

$$A : \mathtt{type} \vdash \mathtt{data}\ \mathtt{Id}(a\ b : A)\ \mathtt{where}$$
$$\mid \mathtt{refl}\ (a : A) : \mathtt{Id}(a, a)$$

$$P \in \mathtt{Path}_A(M_0, M_1)$$
$$\Downarrow$$
$$\mathtt{coe}^{0 \rightsquigarrow 1}_{x.\mathtt{Id}(A)(M_0, P@x)}(\mathtt{refl}(M_0)) \longmapsto \text{“}\mathtt{fcoe}^{0 \rightsquigarrow 1}_{x.(M_0, P@x)}(\mathtt{refl}(M_0))\text{”}$$

$$\mathtt{fcoe}^{r \rightsquigarrow s}_{x.(-,-)}(-) \quad + \quad \mathtt{coe}^{r \rightsquigarrow s}_{x.A}(-) \quad \Rightarrow \quad \mathtt{coe}^{r \rightsquigarrow s}_{x.\mathtt{Id}_A(M_0, M_1)}(-)$$

all in all

- schema for indexed higher inductive types
  - torus, higher truncations, localizations, etc.
  - identity types
- computational semantics
  - PERs on untyped operational semantics
  - canonicity theorem
- fragment implemented in `redtt` proof assistant

`github.com/RedPRL/redtt`

thank you!

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