Taming Reversals in Cubical Type Theories

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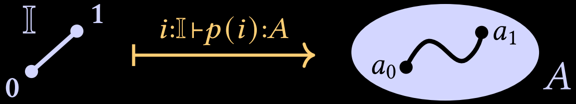
joint work with Christian Sattler

Situation

- Cubical type theory:
  represent identities as terms varying over an abstract interval



- Supports HoTT features w/ good computational properties
- Axiomatize different operations on $\mathbb{I} \implies$ different type theories

  e.g. $i, j : \mathbb{I} \vdash i \wedge j : \mathbb{I}$ $\quad 0 \wedge i = 0 \quad i \wedge (j \vee k) = (i \wedge j) \vee (i \wedge k)$

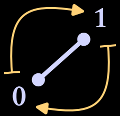
  $i, j : \mathbb{I} \vdash i \vee j : \mathbb{I} \quad 1 \wedge i = i \quad \cdots$

- But hope these "different" theories are equally "good". Would like:
  - Semantics that connect them to "real homotopy theory"
  - Understand conservativity relations between theories

Situation: reversals

- Today, one particular operation: reversal $\neg : \mathbb{I} \to \mathbb{I}$

$$\neg 0 = 1 \qquad \neg\neg i = i$$
$$\neg 1 = 0$$



- Practically useful:

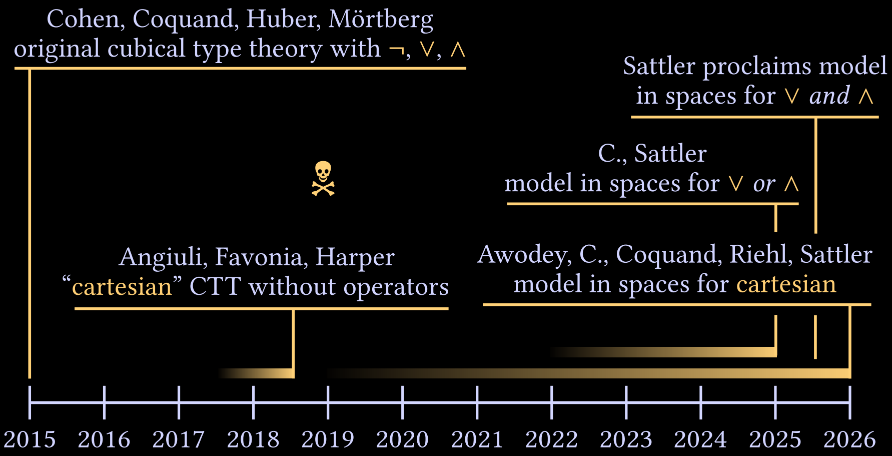
$$\mathsf{sym} : \mathsf{Path}_A(a_0, a_1) \to \mathsf{Path}_A(a_1, a_0)$$
$$\mathsf{sym}(p) := \lambda i.p(\neg i)$$

satisfies $\mathsf{sym}(\mathsf{sym}(p)) = p$
up to judgmental equality

- Concretely: included in Cubical Agda, Mikan proof assistants

- Until now, pain point for well-behaved semantics

Previously...



Can constructions in any CTT be interpreted into topological spaces?

☠ Sattler: known semantics of CTTs have pathological homotopy theories

Now (w/ Sattler)

https://arxiv.org/abs/2605.15080

(coming to LICS 2026)

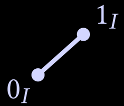
- We show that a reversal ¬ is mostly harmless
  - Method for "upgrading" models of CTT to support ¬
    - Preserves homotopy theory – get model of ¬ in spaces
    - Expect we can apply to Sattler's model with ∧ and ∨ to interpret ≈ Cubical Agda / Mikan
  - focus today { For "opaque" cubical type theory w/ fewer computation rules, prove a conservativity result for extension by ¬
    - Hope: stepping stone to result for full CTT
- Everything is based on one weird trick

I. One Weird Trick

Twist constructions

- Suppose I is a bipointed set, an algebra for the theory of two constants:

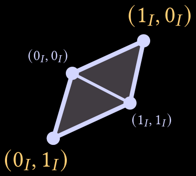
$$0_I \in I \qquad\qquad 1_I \in I$$



- Have a bipointed set I^{op} with same underlying set but $0_{I^{\mathrm{op}}} = 1_I$ and $1_{I^{\mathrm{op}}} = 0_I$

- Take $\mathrm{T}I \coloneqq I \times I^{\mathrm{op}}$, so:

$$0_{\mathrm{T}I} \coloneqq (0_I, 1_I) \qquad\qquad 1_{\mathrm{T}I} \coloneqq (1_I, 0_I)$$



- $\mathrm{T}I$ has a "reversal" $\neg : \mathrm{T}I \cong (\mathrm{T}I)^{\mathrm{op}}$ that permutes coordinates:

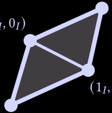
$$\neg(i, j) \coloneqq (j, i)$$

- Suppose I is a distributive lattice:

$$0_I \in I \qquad \vee_I : I \longrightarrow I \longrightarrow I$$
$$1_I \in I \qquad \wedge_I : I \longrightarrow I \longrightarrow I$$

+ lots of equations

- Have a dual distributive lattice I^{op} with
  $0_{I^{\mathrm{op}}} = 1_I$ and $1_{I^{\mathrm{op}}} = 0_I$ and $\wedge_{I^{\mathrm{op}}} = \vee_I$ and $\vee_{I^{\mathrm{op}}} = \wedge_I$



- Take $\mathrm{T}I \coloneqq I \times I^{\mathrm{op}}$, so, e.g.:

$$(i_0, i_1) \vee_{\mathrm{T}I} (j_0, j_1) \coloneqq (i_0 \vee_I j_0, i_1 \wedge_I j_1)$$

- This distributive lattice is a De Morgan algebra

$$\neg(i, j) \coloneqq (j, i)$$

Twist constructions

- More generally, any "self-dual" algebraic theory will support such a T
- Such constructions appear in e.g. order theory, logic literature
- We borrowed the name "twist construction" which originates with Kracht (1998)
- We extend to a twist translation of CTT's:

$$\mathbb{C}\text{TT}_{\neg} \overset{\text{T}}{\dashrightarrow} \mathbb{C}\text{TT}$$

$$\mathbb{I} \mapsto \mathbb{I} \times \mathbb{I}^{\text{op}}$$

II. Scaffolding

What is a cubical type theory?

- Use Uemura (2021, 2023)'s framework of second order generalized algebraic theories (SOGATs) and representable map categories (RMCs)
- Specify CTT as a SOGAT (following Uemura):

Representable judgments can be hypothesized ↓

Ty	:	$() \Rightarrow \mathsf{Jdg}$	$\Gamma \vdash A\ \mathsf{type}$	
Tm	:	$(\mathsf{A} : \mathsf{Ty}) \Rightarrow \mathsf{RepJdg}$	$\Gamma \vdash a : A$	$(\Gamma, a : A)$
$\mathbb{I}$	:	$() \Rightarrow \mathsf{RepJdg}$	$\Gamma \vdash i : \mathbb{I}$	$(\Gamma, i : \mathbb{I})$
Cof	:	$() \Rightarrow \mathsf{Jdg}$	$\Gamma \vdash \phi\ \mathsf{cof}$	
True	:	$(\phi : \mathsf{Cof}) \Rightarrow \mathsf{RepJdg}$	$\Gamma \vdash \_ : \phi\ \mathsf{true}$	$(\Gamma, \_ : \phi)$

- Continue w/ type formers (Σ, Π, ...), cubical stuff (Path, fill, ...)

Comparing type theories

- SOGAT $\mathbb{T}$ generates a classifying category $\mathbb{C}\text{L}(\mathbb{T})$ (or just $\mathbb{T}$) where objects are compound judgments like

$$(\mathsf{A} : \mathsf{Ty},\ \mathsf{B} : \underbrace{\mathsf{Tm}(\mathsf{A})}_{\text{quantification over rep'ble judgment}} \to \mathsf{Ty},\ \mathsf{a} : \mathsf{Tm}(\mathsf{A}),\ \_ : (\mathsf{a}' : \mathsf{Tm}(\mathsf{A})) \to \underbrace{\mathsf{a} \equiv \mathsf{a}'}_{\text{equation}})$$

- This is a representable map category (RMC): category with finite limits and pushforward along fixed class of "rep'ble maps"
- An RMC functor is one presenting this structure
  - RMC functor from a SOGAT ≃ interpretation of its judgments
- We use RMC functors to compare cubical type theories

Functorial semantics

- Uemura defines models of an RMC categorically:

$$\mathbf{Mod}(\mathbb{T}) \coloneqq \textstyle\sum_{\mathcal{C} \in \mathbf{Cat}} (\mathbb{T} \xrightarrow[\text{RMC}]{} \mathrm{PSh}(\mathcal{C}))$$

- Idea, for RMC coming from SOGAT:
  - category of contexts $\mathcal{C}$
  - interpretation of each judgment in each $\Gamma \in \mathcal{C}$

Interval theories

- Consider the SOGAT $\mathbb{I}\textsc{nt}$ of a lonely interval

$$\begin{aligned} \mathbb{I} &: () \Rightarrow \mathrm{RepJdg} \\ 0 &: () \Rightarrow \mathbb{I} \\ 1 &: () \Rightarrow \mathbb{I} \end{aligned}$$

- An interval theory is an object $\Phi \in \mathbb{I}\textsc{nt}$ of the induced RMC

$$\text{e.g.} \qquad \begin{aligned} (\wedge &: (\mathtt{i} : \mathbb{I}, \mathtt{j} : \mathbb{I}) \to \mathbb{I}, \\ \_ &: (\mathtt{i} : \mathbb{I}) \to 0 \wedge \mathtt{i} \equiv 0 : \mathbb{I}, \\ \ldots) \end{aligned}$$

- Appending an interval theory gives extended SOGATs/RMCs $\mathbb{I}\textsc{nt}[\Phi]$ and $\mathbb{C}\textsc{tt}[\Phi]$

Self-dual interval theories

- Have Flip : $\mathbb{I}\text{NT} \to \mathbb{I}\text{NT}$ defined by $\mathbb{I} \mapsto \mathbb{I}$, $0 \mapsto 1$, $1 \mapsto 0$

- A self-duality of an interval theory $\Phi \in \mathbb{I}\text{NT}$ is a $\theta : \text{Flip}(\Phi) \cong \Phi$ such that $\text{Flip}(\theta) \circ \theta = \text{id}$

- A self-dual interval theory (Φ, θ) has an extension by a reversal $\text{Rev}_\theta\Phi \in \mathbb{I}\text{NT}$, which adds to Φ:

$$
\begin{aligned}
\neg &: \mathbb{I} \to \mathbb{I} \\
\_ &: \neg 0 \equiv 1 : \mathbb{I} \\
\_ &: \neg 1 \equiv 0 : \mathbb{I} \\
\_ &: (\mathtt{i} : \mathbb{I}) \to \neg\neg \mathtt{i} \equiv \mathtt{i} : \mathbb{I} \\
\_{\mathtt{c}} &: (\mathtt{i}_1, \ldots, \mathtt{i}_n : \mathbb{I}) \to \neg \mathtt{c}(\mathtt{i}_1, \ldots, \mathtt{i}_n) \equiv \theta(\mathtt{c})(\neg \mathtt{i}_1, \ldots, \neg \mathtt{i}_n) : \mathbb{I} \\
&\qquad \text{for each } \mathtt{c} : (\mathtt{i}_1, \ldots, \mathtt{i}_n : \mathbb{I}) \to \mathbb{I} \text{ in } \Phi
\end{aligned}
$$

III. Twist translation

Twist translation

- Twist construction as a functor $\mathrm{T}\colon \mathbb{I}\textsc{nt}[\mathrm{Rev}_\theta \Phi] \longrightarrow \mathbb{I}\textsc{nt}[\Phi]$ induced by interpretation of the SOGAT:

$$
\begin{aligned}
\mathrm{T}\mathbb{I} &\coloneqq (\mathtt{i}_0 : \mathbb{I}, \mathtt{i}_1 : \mathbb{I}) \\
\mathrm{T}0 &\coloneqq (0, 1) \\
\mathrm{T}1 &\coloneqq (1, 0) \\
\mathrm{T}\neg(\mathtt{i}_0, \mathtt{i}_1) &\coloneqq (\mathtt{i}_1, \mathtt{i}_0) \\
\mathrm{T}\mathtt{c}((\mathtt{i}_0, \mathtt{i}_1), \ldots) &\coloneqq (\mathtt{c}(\mathtt{i}_0, \ldots), \theta(\mathtt{c})(\mathtt{i}_1, \ldots))
\end{aligned}
$$

- Let's extend to a functor $\mathrm{T}\colon \mathbb{C}\textsc{tt}[\mathrm{Rev}_\theta \Phi] \longrightarrow \mathbb{C}\textsc{tt}[\Phi]$:

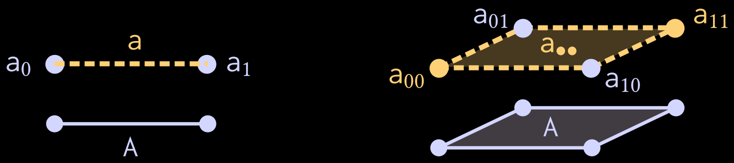
$$
\begin{aligned}
\mathrm{T}\mathsf{Ty} &\coloneqq \mathsf{Ty} \\
\mathrm{T}\mathsf{Tm} &\coloneqq \mathsf{Tm} \\
\mathrm{T}\Sigma &\coloneqq \Sigma \\
\ldots &\coloneqq \ldots
\end{aligned}
$$

- Only cubical features take any work

Twist translation: paths

- Path types internalize paths with fixed endpoints:

$$\begin{aligned} \mathsf{Path} &: (A : \mathbb{I} \to \mathsf{Ty},\ a_0 : A(0),\ a_1 : A(1)) \Rightarrow \mathsf{Ty} \\ \lambda^{\mathbb{I}} &: (A : \mathbb{I} \to \mathsf{Ty},\ a : (i : \mathbb{I}) \to A(i)) \Rightarrow \mathsf{Path}(A, a(0), a(1)) \end{aligned}$$



- Translate to types of squares (iterated paths):

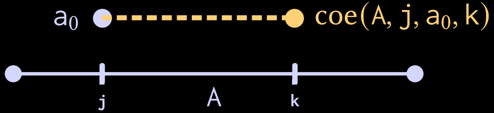
$$(\mathsf{T}\mathsf{Path})(A, a_{01}, a_{10}) :=$$

$$\begin{aligned} &\Sigma a_{00}{:}A(0,0).\ \Sigma a_{11}{:}A(1,1). \\ &\Sigma a_{\bullet 0}{:}\mathsf{Path}(\langle i_0\rangle A(i_0,0), a_{00}, a_{10}).\ \Sigma a_{\bullet 1}{:}\mathsf{Path}(\langle i_0\rangle A(i_0,1), a_{01}, a_{11}). \\ &\Sigma a_{0\bullet}{:}\mathsf{Path}(\langle i_1\rangle A(0,i_1), a_{00}, a_{01}).\ \Sigma a_{1\bullet}{:}\mathsf{Path}(\langle i_1\rangle A(1,i_1), a_{10}, a_{11}). \\ &\mathsf{Path}(\langle i_0\rangle \mathsf{Path}(\langle i_1\rangle A(i_0,i_1), a_{\bullet 0}\ i_0, a_{\bullet 1}\ i_0), a_{0\bullet}, a_{1\bullet}) \end{aligned}$$

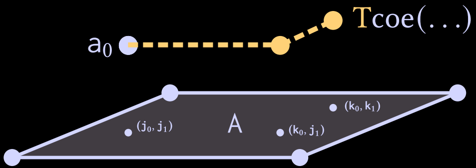
Twist translation: coercion

- Coercion makes the interval "contractible":

$$\mathsf{coe} : (\mathsf{A} : \mathbb{I} \longrightarrow \mathsf{Ty},\ \ \mathsf{j} : \mathbb{I},\ \ \mathsf{a}_0 : \mathsf{A}(\mathsf{j}),\ \ \mathsf{k} : \mathbb{I}) \Rightarrow \mathsf{A}(\mathsf{k})$$



- To define (Tcoe)(A, (j₀, j₁), a₀, (k₀, k₁)), coerce twice



- Does not interpret strict rules for reducing at concrete A

IV. Conservativity

Weak equivalences

- Kapulkin and Lumsdaine (2018) define weak equivalence of democratic[1] models of $\mathbb{M}\,\mathrm{LTT}_{\Sigma,\mathrm{Id}}$

Def: A morphism of models $F\colon \mathcal{M} \to \mathcal{N}$ is a weak equivalence when we have

Weak type lifting: Weak term lifting:

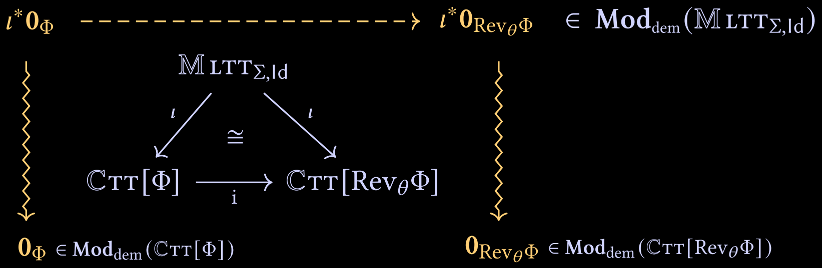
$$
\begin{array}{ccccccccccccccc}
\mathcal{M} \ni & & \Gamma & \vdash & & & A & & \Gamma & \vdash & & & a & : & A \\
\downarrow F & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathcal{N} \ni & & F\Gamma & \vdash & B & \simeq & FA & & F\Gamma & \vdash & b & \asymp & Fa & : & FA
\end{array}
$$

- For $F\colon \mathcal{M} \to \mathcal{N}$ looking like inclusion between syntax-y models, a kind of conservativity

[1]Every context can be made of types

Interpretations to morphisms

- Inclusion $\mathrm{i}\colon \mathbb{C}\textsc{tt}[\Phi] \to \mathbb{C}\textsc{tt}[\mathrm{Rev}_\theta\Phi]$ induces morphism of models in $\mathbb{M}\,\textsc{ltt}_{\Sigma,\mathrm{Id}}$



- $\mathbf{0}_\Phi$: types and terms from $\mathbb{C}\textsc{tt}[\Phi]$, contexts of term variables
- Want to show $\iota^*\mathbf{0}_\Phi \to \iota^*\mathbf{0}_{\mathrm{Rev}_\theta\Phi}$ a weak equivalence: over a term context of $\mathbb{C}\textsc{tt}[\Phi]$, can approximate types/terms of $\mathbb{C}\textsc{tt}[\mathrm{Rev}_\theta\Phi]$ with those of $\mathbb{C}\textsc{tt}[\Phi]$

How to do it

- **Thm:** whenever we have

$$\begin{array}{ccc} & \mathbb{M}\textsc{ltt}_{\Sigma,\mathsf{Id}} + \mathsf{Cof} + \mathsf{True} + \mathsf{U} & \\ {}^{\iota}\swarrow & \cong & \searrow^{\iota} \\ \mathbb{C}\textsc{tt}[\Phi] & \underset{G}{\overset{F}{\rightleftarrows}}\ \# & \mathbb{C}\textsc{tt}[\Psi] \end{array}$$

  the induced maps $\iota^*\mathbf{0}_\Phi \leftrightarrows \iota^*\mathbf{0}_\Psi$ are weak equivalences.

- Idea: up to homotopy, a translation does not have much choice
  - Connectives characterized by universal property up to equivalence (except U)
  - From perspective of types and terms, $\mathbb{I}$ too small to be seen
- See paper for details (involves RMC of spans, …)

We did it

- **Thm:** For an interval theory Φ with a self-duality θ, we have

$$\begin{array}{ccc} & \mathbb{M}\text{LTT}_{\Sigma,\mathsf{Id}} + \mathsf{Cof} + \mathsf{True} + \mathsf{U} & \\ \swarrow^{\iota} & \cong & \searrow^{\iota} \\ \mathbb{C}\text{TT}[\Phi] & \underset{\mathrm{T}}{\overset{\mathrm{i}}{\rightleftarrows}}\ \# & \mathbb{C}\text{TT}[\mathrm{Rev}_\theta\Phi] \end{array}$$

  and so $\iota^* \mathbf{0}_\Phi \longrightarrow \iota^* \mathbf{0}_{\mathrm{Rev}_\theta \Phi}$ is a weak equivalence.

- That is, the extension by a reversal is conservative

- Remember, this is “opaque” cubical type theory without
  - judgmental equations reducing coe/fill at concrete types
  - judgmental computation rules for path constructors of HITs

Ending

- Experiment: how much juice can we squeeze out of a weird trick before the LICS deadline?
- My optimistic feeling: much more general conservativity results hold
  - for strict cubical type theory
  - comparing arbitrary interval theories
- Maybe our worked example for the simplest translation you could dream of can be useful for you

THANKS

IV. Concrete Models

Concrete models

- Angiuli, Brunerie, Coquand, Harper, Favonia, and Licata (2021) proved something more general than

  > Let C be a finite product category with an interval object $1 \overset{0}{\underset{1}{\rightrightarrows}} I$. There is a model of cubical type theory in $\mathrm{PSh}(C)$ interpreting the interval theory of all $f : I^n \to I$.

- If $1 \overset{0}{\underset{1}{\rightrightarrows}} I \in C$ is an "ABCHFL setup", then $1 \overset{(0,1)}{\underset{(1,0)}{\rightrightarrows}} I \times I \in C$ is also an ABCHFL setup.

- **Thm:** The setups $1 \overset{0}{\underset{1}{\rightrightarrows}} I \in C$ and $1 \overset{(0,1)}{\underset{(1,0)}{\rightrightarrows}} I \times I \in C$ yield the same class of semantic types $\{p_A : \Gamma.A \to \Gamma\} \subseteq \mathrm{PSh}(C)^{\to}$.

Concrete models in spaces

- **Cor:** The setups $1 \underset{1}{\overset{0}{\rightrightarrows}} I \in C$ and $1 \underset{(1,0)}{\overset{(0,1)}{\rightrightarrows}} I \times I \in C$ yield models of cubical TT describing the same homotopy theory on $\mathrm{PSh}(C)$.

- C.–Sattler '25: ABCHFL model on $C = \square_\vee \coloneqq \{\text{cubes with } \vee\}$ with $1 \underset{1}{\overset{0}{\rightrightarrows}} \square^1$ presents the homotopy theory of spaces

- **Thm:** The ABCHFL model on $C = \square_\vee$ with $1 \underset{(1,0)}{\overset{(0,1)}{\rightrightarrows}} \square^2$ presents the homotopy theory of spaces and interprets a reversal (but no connections)

- Hope: trick should apply to other model constructions, esp. Sattler's model with $\vee$ and $\wedge$