Fitch-style modalities and parametric adjoints

Evan Cavallo
with Daniel Gratzer, G. A. Kavvos, Adrien Guatto, & Lars Birkedal

draft:

https://jozefg.github.io/papers/modalities-and-parametric-adjoints.pdf

“Modalities”

This talk is about designing type theories with **modalities**

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By this I roughly just mean unary connectives:

$\Box A$	A is necessarily true
$\triangleright A$	A is true later
$\lVert A \rVert_{-1}$	A is merely true
$B \longrightarrow A$	A is true conditional on B
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From a type theory design perspective, the challenge comes when these interact strangely with the ambient context:

$$\frac{\cdot \vdash M : A}{\cdot \vdash \mathrm{box}(M) : \Box A} \quad \text{but not} \quad \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{box}(M) : \Box A}$$

Plan

- What do we need for a well-behaved modality?
- Fitch-style modalities—modal types that are right adjoints
- **Improving elimination** in the Fitch-style—
  left adjoints that are parametric right adjoints
- **FitchTT**—multimodal framework for Fitch-style + PRA modalities

Note: substitution

I'll work in a presentation with explicit substitutions:

$$\Delta \vdash \gamma : \Gamma \qquad\qquad \frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash M : A}{\Delta \vdash M[\gamma] : A[\gamma]}$$

and so without named variables:

$$\frac{\Gamma \text{ ctx} \qquad \Gamma \vdash A \text{ type}}{\Gamma.A \text{ ctx}} \qquad\qquad \begin{array}{l} \Gamma.A \vdash \mathrm{p}_A : \Gamma \\ \Gamma.A \vdash \mathrm{v}_A : A[\mathrm{p}_A] \end{array}$$

Modalities in type theory

Let's suppose we have a categorical model in mind:

simple type theory $\mathbb{T}$	category $\mathbb{C}$
$\Delta \vdash \gamma : \Gamma$	$[\![\gamma]\!] : [\![\Delta]\!] \to [\![\Gamma]\!]$
A type	$[\![A]\!] \in \mathbb{C}$
$\Gamma \vdash M : A$	$[\![M]\!] : [\![\Gamma]\!] \to [\![A]\!]$
?	$F : \mathbb{C} \to \mathbb{C}$

What of F can we bring into the type theory?

Modalities in type theory

Formation is easy in simple type theory:

$$\frac{\Gamma \text{ type}}{\bullet\Gamma \text{ type}} \qquad \frac{A \text{ type}}{\circ A \text{ type}} \qquad \frac{A \in \mathbb{C}}{FA \in \mathbb{C}}$$

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A functorial action gives us a kind of joint **intro-elim**:

$$\frac{\Delta \vdash \gamma : \Gamma}{\bullet\Delta \vdash \bullet\gamma : \bullet\Gamma} \qquad \frac{\Gamma \vdash M : A}{\bullet\Gamma \vdash \mathsf{act}(M) : \bigcirc A} \qquad \frac{f : A \to B}{Ff : FA \to FB}$$

But this rule does not hold up under substitution.

$$\frac{\Gamma \vdash M : A}{\bullet\Gamma \vdash \mathsf{act}(M) : \circ A} \qquad \frac{\Delta \vdash \gamma : \bullet\Gamma \qquad \Gamma \vdash M : A}{\Delta \vdash (\mathsf{act}(M))[\gamma] = \mathbf{?} : \circ A}$$

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We would like to write something like

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but this is impossible unless $\Delta = \bullet\Delta'$ and $\gamma = \bullet\gamma'$!

$$\frac{\text{“}\bullet\Delta' \vdash \bullet\gamma' : \bullet\Gamma\text{”} \qquad \Gamma \vdash M : A}{\bullet\Delta' \vdash (\mathsf{act}(M))[\bullet\gamma'] = \mathsf{act}(M[\gamma']) : \circ A}$$

Modalities in type theory

Bierman & de Paiva 2000: Build a substitution into the term.

$$\frac{\Delta \vdash \gamma : \bullet\Gamma \qquad \Gamma \vdash M : A}{\Delta \vdash \mathsf{mod}(M, \gamma) : \bigcirc A}$$

$$\mathsf{mod}(M, \gamma)[\delta] = \mathsf{mod}(M, \gamma \circ \delta)$$

B–dP use this approach for a necessity operator.

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B–dP use this approach for a necessity operator.

A downside: we want to have an equation like

$$\mathrm{mod}(M, \gamma.\text{🔒} \circ \delta) = \mathrm{mod}(M\gamma, \delta)$$

But it's problematic for deciding equality—
when can a substitution be so factored?

Modalities in type theory

Split-context: Separate contexts and substitutions into zones.

Pfenning–Davies 2001; Kavvos 2020; ...

$$(\Theta; \Gamma)\ \mathsf{ctx} \quad \rightsquigarrow \quad F[\![\Theta]\!] \times [\![\Gamma]\!] \in \mathbb{C}$$

$$\frac{\cdot; \Theta \vdash M : A}{\Theta; \Gamma \vdash \mathsf{mod}(M) : \bigcirc A} \qquad \mathsf{mod}(M)[\theta; \gamma] = \mathsf{mod}(M[\cdot; \theta])$$

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Works best when e.g. $FFA = FA$ or similar—otherwise we want more and more zones

Same with multiple modalities

Modalities in type theory

Fitch-style modality (Clouston 2018):
Assume we have a left adjoint to F.

$$G : \mathbb{C} \longrightarrow \mathbb{C}$$

$$\frac{\frac{GA \longrightarrow B}{}}{A \longrightarrow FB}$$

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Note: we use F on types but not contexts

Clouston 2018: $\blacklozenge \dashv \square$, “possibility” adjoint to “necessity”

Bahr—Grathwohl—Møgelberg 2017: $\checkmark \dashv \triangleright$, “tick” adjoint to “later”

Fitch-style modalities

We can derive a mod from act with the unit $\Gamma \vdash \eta : \bullet(\Gamma.\text{🔒})$.

$$\frac{\Gamma.\text{🔒} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \bigcirc A} \quad \rightsquigarrow \quad \frac{\dfrac{\Gamma.\text{🔒} \vdash M : A}{\bullet(\Gamma.\text{🔒}) \vdash \mathsf{act}(M) : \bigcirc A}}{\Gamma \vdash \mathsf{act}(M)[\eta] : \bigcirc A}$$

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η is the "universal obstruction to stability of act":

$$\begin{aligned} \mathsf{act}(M)[\gamma] &= \mathsf{act}(M)[\bullet(\gamma^{\dagger}) \circ \eta] \\ &= \mathsf{act}(M[\gamma^{\dagger}])[\eta] \end{aligned}$$

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Derive act from mod with the counit: $\mathsf{act}(M) := \mathsf{mod}(M[\varepsilon])$

Fitch-style modalities

A useful parallel: **exponentiation**

$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma.B \vdash M : A}{\Delta \vdash \lambda(M)[\gamma] = \lambda(M[\gamma \times B]) : B \to A}$$

We use a left adjoint and its action on substitutions:

$$(-).B \dashv B \to (-)$$

$$\frac{\Gamma.B \vdash \cdots : A}{\Gamma \vdash \cdots : B \to A}$$

$$\frac{\Delta \vdash \gamma : \Gamma}{\Delta.B \vdash \gamma \times B : \Gamma.B}$$

Elimination in the Fitch style

The adjunction justifies an elim inverting the intro:

$$\frac{\Gamma \vdash M : \bigcirc A}{\Gamma.\text{🔒} \vdash \mathrm{unmod}(M) : A}$$

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Same substitution problem!

$$\frac{\Delta \vdash \gamma : \Gamma.🔒 \qquad \Gamma \vdash M : \bigcirc A}{\Delta \vdash \mathsf{unmod}(M)[\gamma] = \,?\, : A}$$

No hope of reducing unless $\gamma = \gamma'.🔒$

Elimination in the Fitch style

Clouston 2018

Clouston–Mannaa–Møgelberg–Pitts–Spitters 2018 (DRA)

Add a weakening step to the rule:

$$\Xi = A_1.\cdots.A_n \left\{ \frac{\Gamma \vdash \Xi \text{ tel} \qquad \Gamma \vdash M : \bigcirc A}{\Gamma.🔒.\Xi \vdash \mathrm{unmod}_\Xi(M) : A} \right.$$

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Inspect a substitution to determine how to substitute:

$$\mathrm{unmod}_{\Xi}(M)[\mathrm{p}_A] = \mathrm{unmod}_{\Xi.A}(M)$$
$$\mathrm{unmod}_{\Xi.A}(M)[\gamma.N] = \mathrm{unmod}_{\Xi}(M)[\gamma]$$
$$\mathrm{unmod}_{.}(M)[\gamma.🔒] = \mathrm{unmod}_{.}(M[\gamma])$$

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Eg.: $\Gamma.\text{🔒}.\text{🔒} \vdash \mathsf{j} : \Gamma.\text{🔒} \quad \rightsquigarrow \quad \mathsf{unmod}_{.}(M)[\mathsf{j}] = ?$

Gratzer–Sterling–Birkedal 2019, MLTT🔒

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Gratzer–Sterling–Birkedal 2019, $\mathrm{MLTT}_{\text{🔒}}$

Or might want to import substitutions from a model
to get an internal language

Elimination in the Fitch style

Gratzer–Kavvos–Nuyts–Birkedal 2020: **MTT**

Use a positive eliminator with **modal hypotheses**, roughly:

$$\frac{\Gamma \vdash M : \bigcirc A \qquad \Gamma.(\bigcirc \mid A) \vdash N : C}{\Gamma \vdash \mathrm{elim}(M, N) : C} \; \left(\frac{\Delta \to (\bigcirc \mid A)}{\overline{\overline{\Delta.🔒 \to A}}}\right)$$

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Works generally for dependent right adjoints,
and for multiple interacting modalities.

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Works generally for dependent right adjoints,
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but weaker and less convenient than the negative rule—
doesn't embed DRA or MLTT$_{🔒}$

Elimination in the Fitch style

First cut: can we repeat the Fitch-style intro trick?

Assume that we have an adjoint *triple* $H \dashv G \dashv F$

$$\frac{HA \longrightarrow B}{\overline{A \longrightarrow GB}} \qquad \frac{\Delta.\text{🔓} \vdash \gamma^{\dagger} : \Gamma}{\Delta \vdash \gamma : \Gamma.\text{🔒}}$$

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Now we can push all substitutions but the unit into unmod:

$$\begin{aligned} \mathrm{unmod}(M)[\eta] &= \mathrm{unmod}(M)[\gamma^{\dagger}.\text{🔒} \circ \eta] \\ &= \mathrm{unmod}(M[\gamma])[\eta] \end{aligned}$$

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Try naively inverting this rule:

$$\frac{\Gamma \vdash F : B \to A}{\Gamma.B \vdash \mathrm{un}\lambda(F) : A}$$

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$$\mathrm{un}\lambda(F)[\gamma] = \mathrm{un}\lambda(F[\gamma'])[\mathrm{id}.N]$$

Obstruction determined
by $\Delta \vdash N : B$

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Obstruction determined by $\Delta \vdash N : B$

Parametric right adjoints

$$
\begin{array}{ccc}
\Delta & \dashrightarrow & \Gamma.B \\
& {\scriptstyle N} \searrow \quad \swarrow {\scriptstyle !_\Gamma \times B} & \\
& B &
\end{array}
$$

Parametric right adjoints

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$$\begin{array}{ccc} \mathrm{dom}(N) \dashrightarrow \Gamma & & N \dashrightarrow\ !_\Gamma \times B \\ & \cong & \\ \text{in } \mathbb{C} & & \text{in } \mathbb{C}/B \end{array}$$

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$$\mathbb{C}/B \;\underset{!_{(-)} \times B}{\overset{\mathrm{dom}}{\underset{\bot}{\rightleftarrows}}}\; \mathbb{C}$$

Parametric right adjoints

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$$\text{in } \mathbb{C} \qquad\qquad \text{in } \mathbb{C}/B$$

Def: $F : \mathbb{C} \to \mathbb{D}$ is a *parametric right adjoint (PRA)* when $F/1 : \mathbb{C}/1 \to \mathbb{D}/F1$ is a right adjoint.

(see Carboni–Johnstone 1995, Weber 2007)

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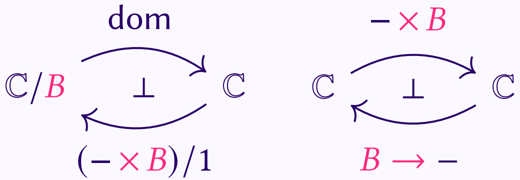

Parametric right adjoints

A less degenerate example: affine functions

Element of $\mathbf{I} \multimap A$ is a term over a "fresh" name

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$$\frac{\overbrace{\Gamma \otimes \mathbf{I}}^{\text{separated product}} \vdash M : A}{\Gamma \vdash \lambda(M) : \mathbf{I} \multimap A}$$

Parametric right adjoints

A less degenerate example: affine functions

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$$\frac{\overbrace{\Gamma \otimes \mathbf{I}}^{\text{separated product}} \vdash M : A}{\Gamma \vdash \lambda(M) : \mathbf{I} \multimap A} \qquad \frac{\Delta \vdash r : \mathbf{I} \qquad \overbrace{\Delta/(r : \mathbf{I})}^{\text{remove } r \text{ from the context}} \vdash F : \mathbf{I} \multimap A}{\Gamma \vdash F@r : A}$$

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A less degenerate example: affine functions

Element of $\mathbf{I} \multimap A$ is a term over a "fresh" name

$$\frac{\overbrace{\Gamma \otimes \mathbf{I}}^{\text{separated product}} \vdash M : A}{\Gamma \vdash \lambda(M) : \mathbf{I} \multimap A} \qquad \frac{\Delta \vdash r : \mathbf{I} \qquad \overbrace{\Delta/(r : \mathbf{I})}^{\text{remove } r \text{ from the context}} \vdash F : \mathbf{I} \multimap A}{\Gamma \vdash F@r : A}$$

$$\lambda(M)@r = M[\eta_r] \qquad F = \lambda(F[\varepsilon_\Gamma]@\mathsf{v})$$

Parametric right adjoints

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see also Cheney 2012, Cavallo–Harper 2020

Parametric right adjoints

It was there all all along!

Recall the DRA rule:

$$\Xi = A_1.\cdots.A_n \left\{ \frac{\Gamma \vdash \Xi\ \mathsf{tel} \qquad \Gamma \vdash M : \bigcirc A}{\Gamma.🔒.\Xi \vdash \mathsf{unmod}_{\Xi}(M) : A} \right.$$

Parametric right adjoints

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Proof of substitution admissibility uses that 🔒 is a PRA *in the syntactic model*

$$\begin{array}{c} \Gamma.🔒.\Xi \\ \downarrow \\ \cdot\,.🔒 \end{array} \quad \longmapsto \quad \Gamma$$

Multimodal type theory

Now we have the key ingredient for a general modal framework, **FitchTT**:

- Parameterized by an arbitrary selection of interacting modalities
  - As in Licata–Shulman 2015, MTT
  - Problematic in earlier Fitch-style calculi
- Elimination of substitution without analysis of substitutions
  - Enables use as an internal language
  - Or addition of substitution axioms more generally (eg. to express properties of the “restriction”)

Mode theory

Licata–Shulman 2015:
Specify interacting modalities with a strict 2-category

m mode	$\rightsquigarrow$	Γ ctx @ m, $\Gamma \vdash A$ type @ $m, \ldots$
$\mu : m \to n$	$\rightsquigarrow$	$\dfrac{\Gamma \text{ ctx @ } m}{\Gamma.\{\mu\} \text{ ctx @ } n}$
$\alpha :: \nu \Rightarrow \mu : m \to n$	$\rightsquigarrow$	$\dfrac{\Gamma \text{ ctx @ } m}{\Gamma.\{\mu\} \vdash \{\alpha\}_\Gamma : \Gamma.\{\nu\} \text{ @ } n}$

Mode theory is a parameter to FitchTT

Mode theory

On top, we make each $-.\{\mu\}$ a PRA:

$$\frac{\mu : n \to m \qquad \Gamma \text{ ctx} @ n \qquad \Gamma \vdash r : \cdot.\{\mu\}}{\Gamma/(r : \mu) \text{ ctx} @ m}$$

Mode theory

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With unit and counit like so:

$$\Gamma \vdash \eta[r] : \Gamma/(r : \mu).\{\mu\} @ n$$

$$\Gamma.\{\mu\}/(!.\{\mu\} : \mu) \vdash \epsilon[\Gamma] : \Gamma @ m$$

and triangle, naturality, etc equations...

Modal types

Formation—matches introduction

$$\frac{\mu : n \to m \qquad \Gamma.\{\mu\} \vdash A \text{ type } @\ n}{\Gamma \vdash \langle \mu \mid A \rangle \text{ type } @\ m}$$

Modal types

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Introduction—Fitch-style

$$\frac{\mu : n \to m \qquad \Gamma.\{\mu\} \vdash M : A\ @\ n}{\Gamma \vdash \mathsf{mod}(M) : \langle \mu \mid A \rangle\ @\ m}$$

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Introduction—Fitch-style

$$\frac{\mu : n \to m \qquad \Gamma.\{\mu\} \vdash M : A @ n}{\Gamma \vdash \mathsf{mod}(M) : \langle \mu \mid A \rangle @ m}$$

Elimination—using parametric right adjoint

$$\frac{\mu : n \to m \qquad \Gamma/(r : \mu) \vdash F : \langle \mu \mid A \rangle @ m}{\Gamma \vdash F@r : A[\eta_r] @ n}$$

For example

To encode affine functions, add transformations for structural rules:

$$\frac{}{m \text{ mode}} \qquad \frac{}{\mu : m \to m}$$

$$\frac{}{\mathit{weak} :: \mathrm{id} \Rightarrow \mu : m \to m}$$

$$\frac{}{\mathit{exch} :: \mu \circ \mu \Rightarrow \mu \circ \mu : m \to m}$$

+ equations

Models

FitchTT is a GAT, so we get a notion of model for free

Moreover, presheaves are a simple source of models:

Thm: Fix a pseudofunctor $F : \mathcal{M}^{coop} \to Cat$ such that $F(m) = PSh(C_m)$ for each $m : \mathcal{M}$, and for each $\mu : n \to m$ either

1. $F(\mu) = f_!$ for a PRA $f : C_m \to C_n$.
2. $F(\mu) = f^*$ for any $f : C_n \to C_m$.

Then there exists a model of FitchTT with mode theory $\mathcal{M}$ where $F(m)$ models mode m and $F(\mu)$ models $-.\{\mu\}$.

Examples

- Affine cubical sets
  - $\mathbf{I} \multimap A$ with $0, 1 \in \mathbf{I}$
  - Bezem–Coquand–Huber 2013,
    Bernardy–Coquand–Moulin 2015
  - Nominal sets by generalizing to sheaves (cf. Cheney 2015)

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- Guarded type theory semantics in $PSh(\omega)$
  - qua stripped-down Clocked Type Theory (CloTT)
    Bahr–Grathwohl–Møgelberg 2017
  - "later" $\triangleright$ and "always" $\square$ with $\square\triangleright A \simeq \square A$
  - See *tick variables*—function-like presentation of later—
    as an instance of PRA structure

Closing

- Assuming a left adjoint—to F or $F / 1$—neatly resolves admissibility of substitution into the right adjoint

- Several existing systems support this FitchTT PRA structure —in both syntactic and standard semantic models

- Get a convenient “variable”-based syntax for such modalities

- In other cases, defer to **MTT**—combine with **FitchTT**?