Replacement in cubical models and type theories

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The type-theoretic axiom of replacement

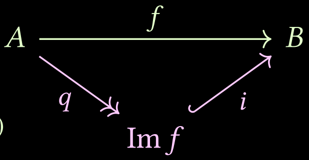
- In MLTT with a universe $\mathcal{U}$:

  Axiom Schema (Replacement). Any map $f : A \to B$ from a $\mathcal{U}$-small type to a locally $\mathcal{U}$-small type has a $\mathcal{U}$-small image

  - $\mathcal{U}$-small = equivalent to a type in $\mathcal{U}$
  - locally $\mathcal{U}$-small = identity types are $\mathcal{U}$-small
  - image = (surjection, embedding) factorization

for all $x : \mathrm{Im}\, f \vdash P\, x$ prop,

$\prod_{x:\mathrm{Im}\, f} P\, x \xrightarrow{\sim} \prod_{a:A} P\, (q\, a)$



$\forall\, x_0, x_1 : \mathrm{Im}\, f,$

$x_0 = x_1 \xrightarrow{\sim} i\, x_0 = i\, x_1$

History

- Rijke 2017, *The join construction*
  - Replacement is a theorem in MLTT with univalent $\mathcal{U}$, global function extensionality, and pushouts

$$f \longrightarrow f *_B f \longrightarrow f *_B f *_B f \longrightarrow \cdots \dashrightarrow i$$

- Rijke 2018, *Classifying types*
  - Connects to replacement axiom schema of set theory

- Rijke 2022, *Introduction to homotopy type theory*
  - First stated and used as an axiom

Examples. I.

- Images in $\mathcal{U}$ (of course)
  - prop truncation $\|A\|_{-1} = \mathrm{Im}(A \to 1)$
- ⚠ More interesting examples will use univalence, global funext
- Higher truncations

$$a \longmapsto (b \mapsto \|a = b\|_{-1})$$

$$A \longrightarrow (A \to \mathcal{U})$$

$$A \twoheadrightarrow \|A\|_0 \hookrightarrow (A \to \mathcal{U})$$

Examples. I.

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$$a \longmapsto (b \mapsto \|a = b\|_n)$$

$$A \longrightarrow (A \to \mathcal{U})$$

$$A \twoheadrightarrow \|A\|_{n+1} \hookrightarrow (A \to \mathcal{U})$$

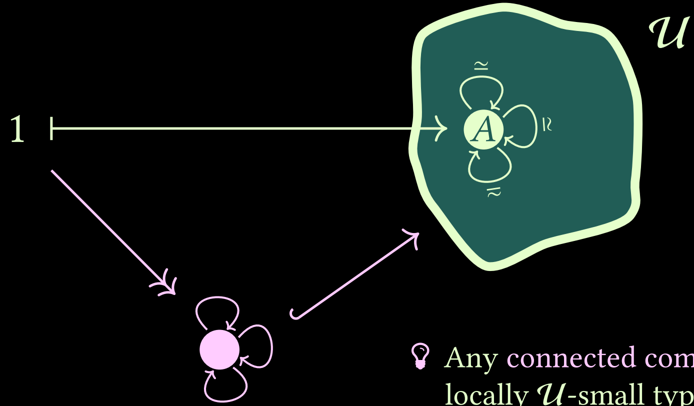
- Set quotients of equivalence relations $R : A \to A \to \mathcal{U}_{prop}$

$$A \longrightarrow (A \to \mathcal{U}_{prop})$$

$$a \longmapsto (b \mapsto R\ a\ b)$$

Examples. II.

- To be an embedding in HoTT is a strong property!



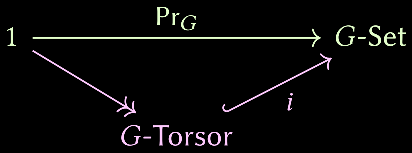
- Any connected component of a locally $\mathcal{U}$-small type is $\mathcal{U}$-small

Examples. III.

- Delooping of a group G via torsors

  $G\text{-Set} = \{X \in \mathcal{U}_{\text{set}} \text{ with an action } \alpha \colon G \times X \to X\}$
  is locally $\mathcal{U}$-small

$\mathrm{Pr}_G : G\text{-Set}$
$\mathrm{Pr}_G = (G, m \colon G \times G \to G)$



$\Omega(G\text{-Torsor}) \cong G$

Examples. IV.

- Two constructions of $K(G,n)$:
  - Buchholtz–Christensen–Flaten–Rijke:

    $$K(G,n+1) = \sum_{Y:\mathcal{U}} \|K(G,n) \simeq Y\|_0$$

  - Wärn:

    $$K(G,n+1) = \sum_{Y:\mathcal{U}} (\|Y\|_{-1} \times \prod_{y:Y} (K(G,n) \simeq_{\mathrm{pt}} (Y,y)))$$

- In each def'n, $Y : \mathcal{U}$ can be restricted to component of $K(G,n)$
- ↑ = instances of more general delooping constructions
- **Corollary:** $\sum_{n:\mathbb{N}} K(G,n)$ is $\mathcal{U}$-small
  - $\mathcal{U}$ contains non-truncated types!

Examples. V.

- Rezk completion of a category
  - Univalent category (isomorphism is equality) from a category, as in HoTT Book (§9.9):

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\quad よ \quad} & [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}_{\mathcal{U}}] \\ & \searrow\!\!\!\twoheadrightarrow \quad \overline{\mathscr{C}} \quad \hookrightarrow\!\nearrow & \end{array}$$

  - Delooping of a group is a special case:
    G-Torsor is image of $よ : 1 \to [G^{\mathrm{op}}, \mathbf{Set}_{\mathcal{U}}]$

Set-theoretic replacement?

- Replacement axiom schema of set theory:
  for any class function F and set A, the image $F[A]$ is a set
  - Not in Zermelo's original set theory (1908);
    proposed by at least Fraenkel (1922) and Skolem (1923)
  - Motivated by transfinite number theory:
    e.g., need replacement to build sets of cardinality $\geq \aleph_\omega$

$$\text{why is } \bigcup\{\beth_0, \beth_1, \ldots\} \text{ a set?}$$

- Used for set quotients in CZF—can use power set in ZF

Set-theoretic replacement?

- Traditional “equivalent” in type theory: universes

> Furthermore, there are axioms for universes (in the sense of category theory) which link the generation of objects and types and play somewhat the same role for the present theory as does the replacement axiom for Zermelo–Fraenkel set theory.
>
> — Martin-Löf 1973

$$\beth_{(-)} : \mathbb{N} \to \mathcal{U} \quad \Longrightarrow \quad \sum_{n:\mathbb{N}} \beth_n : \mathcal{U}$$

Set-theoretic replacement? (Examples. VI.)

- Aczel (1978) models CZF as a setoid in MLTT:

$$\mathcal{V}_\infty = \mathrm{W}_{A:\mathcal{U}} A \qquad \frac{A : \mathcal{U} \qquad f : A \to \mathcal{V}_\infty}{\sup(A, f) : \mathcal{V}_\infty}$$

$$(x \approx y) \; = \; \prod_{a:\mathcal{V}_\infty} (a \in x \iff a \in y)$$

- Gylterud (2018) restricts to “iterative sets” $\mathcal{V}$:

$$\frac{A : \mathcal{U} \qquad f : A \hookrightarrow \mathcal{V}}{\sup(A, f) : \mathcal{V}}$$

  - in HoTT, get $(x \approx y) \simeq (x = y)$
  - to interpret replacement, use replacement

Interpreting replacement

- Can we justify replacement directly in e.g., cubical models?
- Can be inspired by case of prop truncation:
  - As instance of join construction:

$$A \longrightarrow A * A \longrightarrow A * A * A \longrightarrow \cdots \dashrightarrow \|A\|_{-1}$$

  - As recursive HIT:

$$\mathtt{in} : A \to \|A\|_{-1} \qquad \mathtt{tr} : \prod_{x,y:\|A\|_{-1}} x = y$$

  - Convenient semantics in fibrant cubical sets (Thierry):

$$\frac{a \in A}{\mathtt{in}(a) \in \|A\|_{-1}} \qquad \frac{x_0 \in \|A\|_{-1} \qquad \phi \vdash x \in \|A\|_{-1}}{\mathtt{ext}(x_0, \phi \mapsto x) \in \|A\|_{-1}\ [\phi \mapsto x]}$$

used in Cherubini–Coquand–Hutzler (2023), Bocquet (2023)

Interpreting replacement: recursive description

- Given $f : A \to B$ with $A : \mathcal{U}$ and B locally $\mathcal{U}$-small, think of

$$A \xrightarrow{\mathsf{in}} \operatorname{Im} f \xrightarrow{\mathsf{out}} B$$

  as generated by

$$\begin{aligned} \mathsf{in} &: A \to \operatorname{Im} f \\ \mathsf{eq} &: \prod_{x,y:\operatorname{Im} f} \mathsf{out}\ x =_B \mathsf{out}\ y \to x =_{\operatorname{Im} f} y \end{aligned}$$

  where

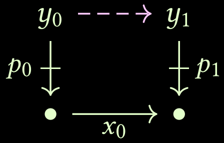
$$\begin{aligned} \mathsf{out}\ (\mathsf{in}\ a) &= f\ b \\ \mathsf{out}\ (\mathsf{eq}\ x\ y\ p) &= p \end{aligned}$$

- Looks like an inductive-recursive definition

Interpreting replacement: cubical sets

$$\frac{a \in A}{\mathtt{in}(a) \in \mathtt{Im}\, f} \qquad \frac{x_0 \in \mathtt{Im}\, f \qquad \phi \vdash w \in \sum_{x:\mathtt{Im}\, f} \mathtt{out}(x) =_B \mathtt{out}(x_0)}{\mathtt{ext}(x_0; \phi \mapsto w) \in \mathtt{Im}\, f\ [\phi \mapsto w.1]}$$

$$\mathtt{ext}\left(x_0;\ \begin{array}{l} i = 0 \mapsto (y_0, p_0) \\ i = 1 \mapsto (y_1, p_1) \end{array}\right)$$

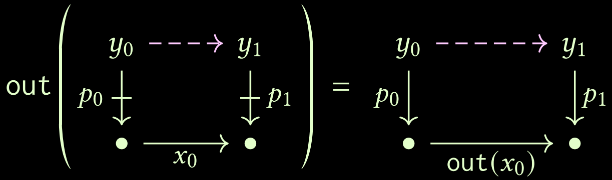


- Reminiscent of Glue types in univalent universe. Simultaneously
  - turn paths in B into paths in $\mathtt{Im}\, f$,
  - implement (homogeneous) composition in $\mathtt{Im}\, f$

Interpreting replacement: cubical sets

$$\frac{a \in A}{\mathtt{in}(a) \in \mathtt{Im}\, f} \qquad \frac{x_0 \in \mathtt{Im}\, f \qquad \phi \vdash w \in \sum_{x:\mathtt{Im}\, f} \mathtt{out}(x) =_B \mathtt{out}(x_0)}{\mathtt{ext}(x_0; \phi \mapsto w) \in \mathtt{Im}\, f\ [\phi \mapsto w.1]}$$

$$\mathtt{out}(\mathtt{in}(a)) = f\ a$$

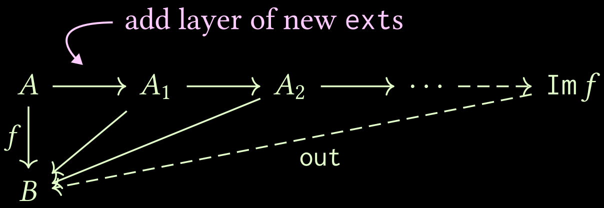


- $\sum_{x:X} \mathtt{out}(x_0) = \mathtt{out}(x)$ are propositions $\implies$ out an embedding
- Can show in surjective by cases

$$x : \mathtt{Im}\, f \vdash P\ x \text{ prop} \qquad \mathtt{elim}_P(f, \mathtt{in}(a)) = f\ a$$

$$f : \prod_{a:A} P\ a \qquad \mathtt{elim}_P(f, \mathtt{ext}(x_0, \phi \mapsto w)) = \cdots$$

- Was induction-recursion necessary? No, can stage:



  - ☆ Why? Maybe: $\mathtt{out} : \mathtt{Im}\, f \to B$ lands in a large type (danger!), but $\mathtt{Im}\, f$ only mentions the small identity types

- Did we use replacement in the metatheory?
  - ☆ Hard to keep track, but at least need $\mathrm{colim}_{n\in\mathbb{N}} A_n$

Closing

What is the strength of replacement relative to pushouts?

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