# Internally Parametric Cubical Type Theory

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dimension context

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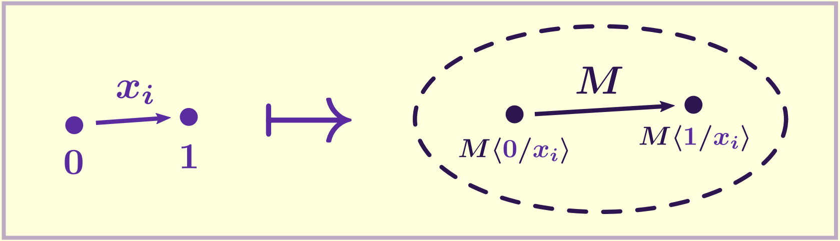

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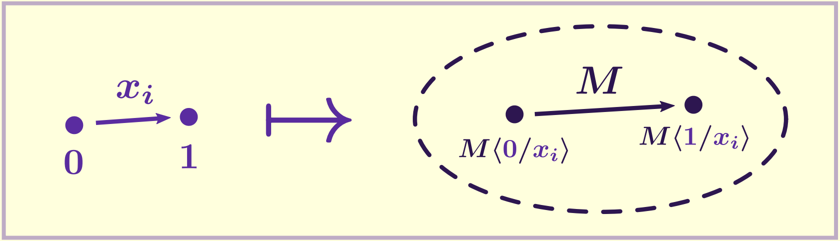


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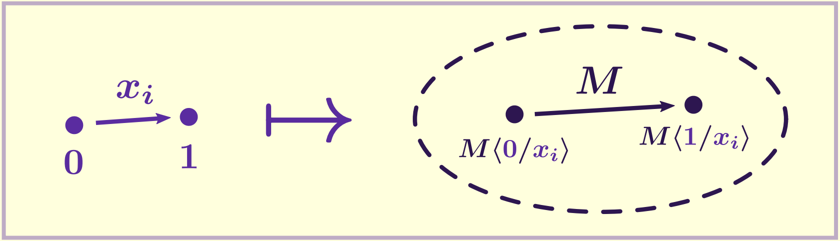


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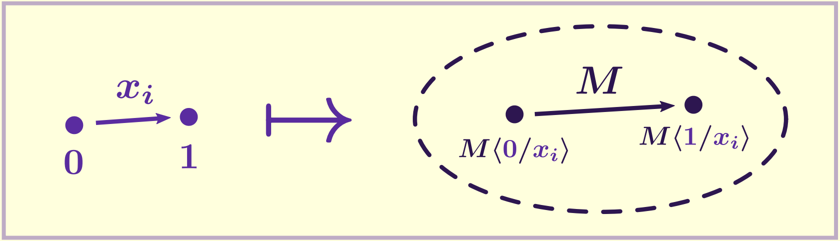


$$\lambda x_i.M \in \mathsf{Path}_{x_i.A}(M\langle 0/x_i\rangle, M\langle 1/x_i\rangle)$$

- **coercion** operation ensures everything respects paths
- **univalence:** type paths are isomorphisms

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} no contraction (diagonals)

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    **data int where**

    $\mid \textbf{neg}(n : \textbf{nat}) : \textbf{int}$

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    $\mid \textbf{seg}(x : \mathbb{I}) : \textbf{int}\ [x = 0 \hookrightarrow \textbf{neg}(0) \mid x = 1 \hookrightarrow \textbf{pos}(0)]$

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    **data circle where**

    | $\mathbf{base} : \mathbf{circle}$

    | $\mathbf{loop}(x : \mathbb{I}) : \mathbf{circle}\ [x = 0 \hookrightarrow \mathbf{base} \mid x = 1 \hookrightarrow \mathbf{base}]$

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(van Doorn 2018, Brunerie 2018)

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- Compare with “ad-hoc” polymorphic functions:

$$\lambda a.\left[\begin{array}{ll} \textbf{true}, & \text{if } X = \textbf{bool} \\ a, & \text{otherwise} \end{array}\right] \in X \to X$$

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$$F_X \in X \to X :$$

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$$R(a, b) \text{ implies } R(F_A(a), F_B(b))$$

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- **Abstraction theorem**: the denotation of any term in simply-typed λ-calculus (with ×, bool) is parametric.
- Key idea: λ-calculus has a **relational interpretation**.

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$$\Downarrow$$

$$F_A(a) = a$$

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| **univalence:** $\textbf{Path}_{\mathcal{U}}(A, B) \simeq (A \simeq B)$ | **relativity:** $\textbf{Bridge}_{\mathcal{U}}(A, B) \simeq A \times B \to \mathcal{U}$ |

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(arXiv:1901.00489)

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**structural**

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(difference invisible for paths because of coercion)

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④ Bridge-discrete types

if $(a_0, a_1 : A) \to \textsf{Bridge}_A(a_0, a_1) \simeq \textsf{Path}_A(a_0, a_1)$,

then $A \simeq (X : \mathcal{U}) \to (A \to X) \to X$

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