Interpreting cubical types as spaces

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Cubical type theory

- HoTT: MLTT with univalence axiom
  - $(A =_{\mathcal{U}} B) \simeq (A \simeq B)$
- HoTT lacks *canonicity*
  - $\cdot \vdash N : \mathbb{N} \;\not\Rightarrow\; N = \text{a numeral} : \mathbb{N}$
- What to do?
  - accept **homotopy canonicity**: get a path $P : N =_{\mathbb{N}}$ a numeral [Kapulkin–Sattler '??, Bocquet '23]
  - concoct an **interpretation where we can compute**
  - build a **new type theory** with canonicity
    - cubical type theories
    - H.O.T.T [Altenkirch–Kaposi–Shulman]

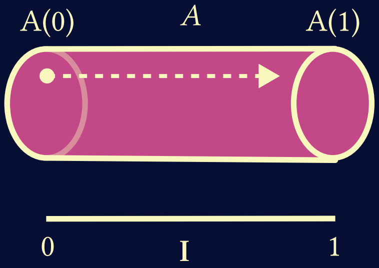
Cubical type theory

⊗ Axiomatize an **interval** (or **cylinder**)

$$\frac{\Gamma \text{ ctx}}{\Gamma, i : \mathbf{I} \text{ ctx}} \qquad \text{path in } \Gamma \vdash A \quad \leftrightarrow \quad \text{element of } \Gamma, i : \mathbf{I} \vdash A$$

⊗ make "everything respect equality" w/ **Kan operations** with computation rules at each type

$$\frac{\Gamma, i : \mathbf{I} \vdash A(i) \text{ type} \qquad \Gamma \vdash M : A(\epsilon)}{\Gamma, i : \mathbf{I} \vdash \text{coe}_A^{\epsilon \to i}(M) : A(i)}$$



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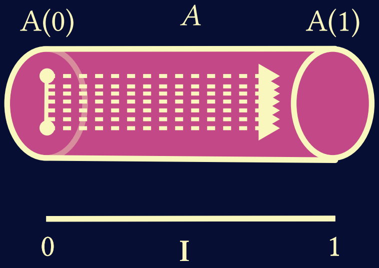

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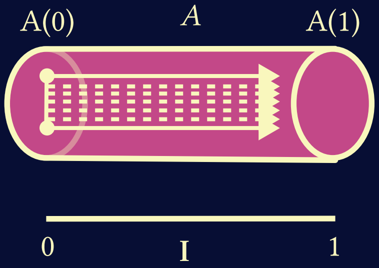
at path types ~ box filling



Variations

- Can impose additional structure on the interval

$$\frac{\Gamma \vdash i : \mathbf{I} \qquad \Gamma \vdash j : \mathbf{I}}{\Gamma \vdash i \vee j : \mathbf{I}} \qquad\qquad \frac{\Gamma \vdash i : \mathbf{I}}{\Gamma \vdash 1 - i : \mathbf{I}}$$

“max” **connection** — reversal

- $\Gamma, i : \mathbf{I}$ can behave like cartesian product (most cubical tt’s), but doesn’t have to [Bezem, Coquand, Huber ’13]

- Convenience feature for users (e.g. $(p^{-1})^{-1} = p$), can also simplify/complicate implementation

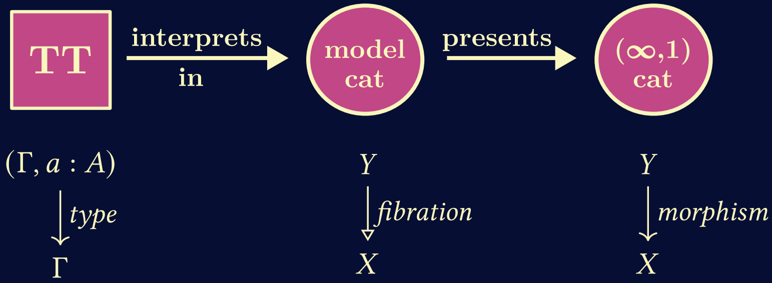
Applying cubical type theory

- ⊗ Advantage of “structural“ theories: many interpretations
  - ⊘ Extensional TT with $\Pi, \Sigma, \mathsf{Id} \sim$ LCCC’s
    [Seely] [Hofmann]
  - ⊘ Intensional TT with $\Sigma, \mathsf{Id} \sim (\infty, 1)$-cats w/ finite limits
    [Kapulkin–Szumilo ’19]
  - ⊘ HoTT $\longrightarrow (\infty, 1)$-topos [Shulman ’19]

- ⊗ Where can we interpret cubical type theories?
  - ⊘ want to relate known models to $(\infty, 1)$-categories
  - ⊘ especially want interpretation in ∞-groupoids
  - ⊘ does choice of $\mathbf{I}$ matter?

Cubical type theory and model categories

⊗ Factor relation to an $(\infty, 1)$-cat thru **model category** presentation



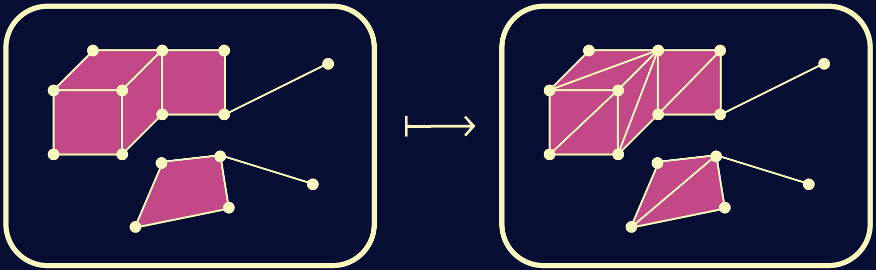
⊗ Interpretations yield model structures on cats of **cubical sets**

- ⊘ with connections $\vee, \wedge$ [Sattler '17]
- ⊘ with cartesian cylinder [C–Mörtberg–Swan '20] [Awodey '23]

Interpreting in ∞-groupoids

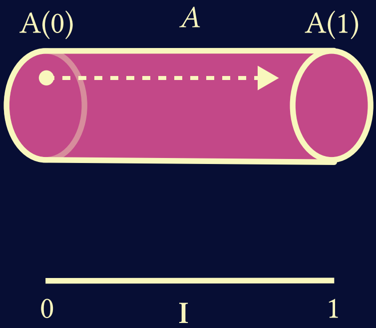
- To compare w/ ∞-groupoids, look for **Quillen equivalence** with classical model structure on simplicial sets

$$\widehat{\square} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \widehat{\Delta}$$



Cartesian cubical sets

⊗ [Awodey–C–Coquand–Riehl–Sattler ’24]
A modification to the model in cartesian cubical sets is needed



⊗ [Awodey–C–Coquand–Riehl–Sattler ’24]
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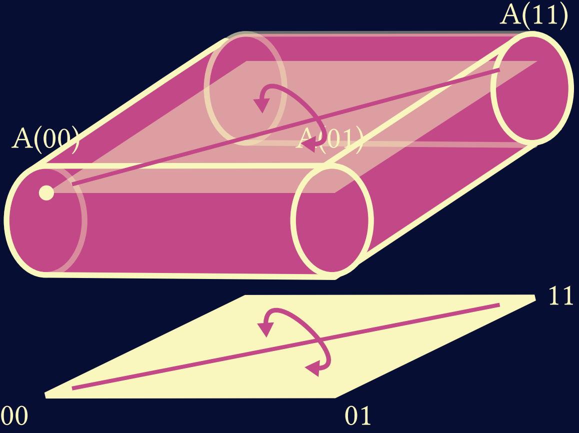
coercion wrt
n-**cubes** of types

equivariant in
the symmetries
of the cube

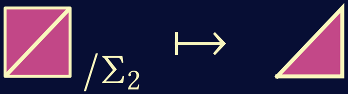


Cartesian cubical sets

- Cartesian cubical sets can* be built by attaching quotients of cubes by symmetries

$$(\partial \mathbf{I}^n)/G \rightarrowtail \mathbf{I}^n/G$$

- Triangulation sends these to contractible simplicial sets



- Key: equivariant fibration model makes them contractible in cubical sets

Cartesian cubical sets with one connection

- [C–Sattler ’22]
- Here, symmetry quotients are easily contractible

 $_{/\Sigma_2} \times \mathbf{I} \longrightarrow$  $_{/\Sigma_2}$

$$([i, j], t) \mapsto [i \vee t, j \vee t]$$

- Equivariant lifting actually derivable from “ordinary” lifting
- But cubical sets no longer made by attaching $(\partial\mathbf{I}^n)/G \rightarrowtail \mathbf{I}^n/G$
  - Cart. cube cat with $\vee$ is not a **generalized Reedy category**
  - Had to invent new generalization to deal with these
  - New generators: more quotients $(\partial\mathbf{I}^n)/R \rightarrowtail \mathbf{I}^n/R$

Using a modality

- Ongoing work by Coquand, Höfer, Sattler in Göteborg

  https://www.cse.chalmers.se/~sattler/docs/external-lex-operation-intuition.pdf
  Towards Poset Type Theory @ TYPES 2024

- When $i \colon \Delta \hookrightarrow \square$, get a monad from induced adjunction

$$\widehat{\square} \underset{i_*}{\overset{i^*}{\underset{\bot}{\rightleftarrows}}} \widehat{\Delta}_+$$

- By theory of modalities [Rijke–Shulman–Spitters '20] [Coquand–Ruch–Sattler '21], can relativize model to **modal types**—those where $\eta_A \colon A \longrightarrow i_* i^* A$ is equivalence

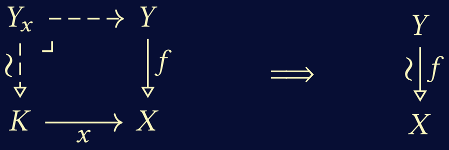
- “Cubical sets that are determined by their simplices”

Which is best?

- Classically:
  - Equivariant model: combinatorially simplest cubical sets
  - One connection model: easiest to write down
  - Relativized model: “most general” construction
- **Constructively?**
  - We don’t know if these models are all equivalent!
  - Much uncertainty in constructive homotopy theory ([Shulman ’21] discusses)
  - We know the relativized model has some good properties: dependent choice, Whitehead’s principle, …

Some ongoing work

- ⊗ How to show that a model does **not** present spaces?
  - ⊘ Christian Sattler came up with arguments in 2018, we're writing these down now
  - ⊘ Idea: identify model-categorical invariants that hold of ∞-groupoids (see my HoTTEST 2024 talk)



for all $K \xrightarrow{\sim} 1$, $x : K \to X$

- ⊗ Haven't thought constructively about this yet!

Some ongoing work

- ⊗ What $(\infty, 1)$-categories do other models present?
  - ⊘ Think we (Tim Hosgood, Reid Barton, I) can show the non-equivariant model structure on cartesian cubical sets presents a presheaf $(\infty, 1)$-category using [Montaruli '22]
- ⊗ Can we build an equivariant fibration model in affine (BCH) cubical sets?
  - ⊘ I think that if it exists, it presents ∞-groupoids
  - ⊘ But interpretation of function types is not as simple to extend...
- ⊗ Is there a model presenting ∞-groupoids with reversals?
  - ⊘ Hope to adapt equivariance to this case

Some decidedly non-ongoing work

- Is there a cubical type theory we can interpret in any ∞-topos?
- Can we prove conservativity results between cubical type theories?
- What happens in cartesian cubical sets with $\vee$ and $\wedge$??

thank you